

RELATIONS BETWEEN BLOCKING SETS AND \mathfrak{C}_n -GROUPS

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ABSTRACT. A blocking set in $PG(n, q)$ is a set of points that has nonempty intersection with every hyperplane of $PG(n, q)$. A blocking set that contains a line is called trivial. A blocking set is called minimal if none of its proper subsets are blocking sets. A cover \mathcal{C} for a group G is called a \mathfrak{C}_n -cover whenever \mathcal{C} is an irredundant maximal core-free n -cover for G and in this case we say that G is a \mathfrak{C}_n -group.

In this paper we give relations between blocking sets and \mathfrak{C}_n -groups.

1. INTRODUCTION

Let G be a group. A set \mathcal{C} of proper subgroups of G is called a cover for G if its set-theoretic union is equal to G . If the size of \mathcal{C} is n , we call \mathcal{C} an n -cover for the group G . A cover \mathcal{C} for a group G is called irredundant if no proper subset of \mathcal{C} is a cover for G . A cover \mathcal{C} for a group G is called core-free if the intersection $D = \bigcap_{M \in \mathcal{C}} M$ of \mathcal{C} is core-free in G , i.e. $D_G = \bigcap_{g \in G} g^{-1} D g$ is the trivial subgroup of G . A cover \mathcal{C} for a group G is called maximal if all the members of \mathcal{C} are maximal subgroups of G . A cover \mathcal{C} for a group G is called a \mathfrak{C}_n -cover whenever \mathcal{C} is an irredundant maximal core-free n -cover for G and in this case we say that G is a \mathfrak{C}_n -group.

A blocking set in $PG(n, q)$ is a set of points that has nonempty intersection with every hyperplane of $PG(n, q)$. A blocking set that contains a line is called trivial. A blocking set is called minimal if none of its proper subsets are blocking sets. For a blocking set B in $PG(n, q)$ we denote by $d(B)$ the least positive integer d such that B is contained in a d -dimensional subspace of $PG(n, q)$. Thus $d(B)$ is equal to the (projective) dimension of the subspace $\langle B \rangle$ in $PG(n, q)$.

Nontrivial minimal blocking sets in $PG(2, p)$ of size $\frac{3(p+1)}{2}$ exist for all odd primes p . Indeed, an example is given by the projective triangle: the set consisting of the points $(0, 1, -s^2)$, $(1, -s^2, 0)$, $(-s^2, 0, 1)$ with $s \in \mathbb{F}_p$.

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For further studies in the topic of blocking sets see Chapter 13 of the second edition of Hirschfeld's book [4] and also see [3] and for further studies in the topic of covering groups by subgroups see [1] and [2].

In this paper we give relations between blocking sets and \mathfrak{C}_n -groups.

2. MAIN RESULTS

As we mentioned in the last section, by a blocking set in $PG(n, q)$, we mean a blocking set with respect to hyperplanes in $PG(n, q)$.

Now we give some notations and definitions as needed in the sequel. We denote the product of n copies of \mathbb{F}_q by $(\mathbb{F}_q)^n$. We note that $(\mathbb{F}_q)^n$ is a vector space of dimension n over \mathbb{F}_q . If $b = (b_1, \dots, b_n) \in (\mathbb{F}_q)^n$, we denote by M_b the set of elements $x = (x_1, \dots, x_n) \in (\mathbb{F}_q)^n$ such that $b \cdot x = \sum_{i=1}^n b_i x_i$ is equal to zero. Note that if $0 \neq b$, then M_b is an $(n-1)$ -dimensional subspace of the vector space $(\mathbb{F}_q)^n$ and every $(n-1)$ -dimensional subspace of $(\mathbb{F}_q)^n$ equals to M_b for some non-zero $b \in (\mathbb{F}_q)^n$. Since for every $0 \neq \lambda \in \mathbb{F}_q$, $M_b = M_{\lambda b}$, $M_{\mathbf{p}}$ is well-defined for every point \mathbf{p} of $PG(n-1, q)$, and $M_{\mathbf{p}}$ may be considered as a hyperplane in $PG(n-1, q)$. We now give some results which clarify the relations between non-trivial minimal blocking sets of size n and \mathfrak{C}_n -covers for groups.

Proposition 2.1. *Let B be a set of points in $PG(n, q)$. Then B is a blocking set in $PG(n, q)$ if and only if the set $\mathcal{C} = \{M_b \mid b \in B\}$ is a $|B|$ -cover for the abelian group $(\mathbb{F}_q)^{n+1}$.*

Proposition 2.2. *Let B be a set of points in $PG(n, q)$. Then B is a minimal blocking set in $PG(n, q)$ if and only if the set $\mathcal{C} = \{M_b \mid b \in B\}$ is an irredundant $|B|$ -cover for the abelian group $(\mathbb{F}_q)^{n+1}$.*

Remark 2.3. *Note that if q is prime, then the cover \mathcal{C} in the statements of Propositions 2.1 and 2.2 is a maximal cover for $(\mathbb{F}_q)^{n+1}$.*

Remark 2.4. *It is easy to see that a (minimal) blocking set B with $d(B) = d$ in $PG(n, q)$ can be obtained from a (minimal) blocking set in $PG(d, q)$. So if we adopt an induction process on n to find all minimal blocking sets B in $PG(n, q)$, we must find only all those minimal blocking sets with $d(B) = n$.*

Proposition 2.5. *Let B be a set of points in $PG(n, q)$. Then B is a blocking set with $d(B) = n$ if and only if the set $\mathcal{C} = \{M_b \mid b \in B\}$ is a core-free $|B|$ -cover for the abelian group $(\mathbb{F}_q)^{n+1}$.*

Theorem 2.6. *Let p be a prime number and n be a positive integer. Then a finite p -group G admits a \mathfrak{C}_{n+1} -cover if and only if $G \cong (C_p)^{m+1}$ for some positive integer m such that $PG(m, p)$ has a minimal blocking set B with $d(B) = m$ and $|B| = n + 1$.*

Proof. Let G be a finite p -group admitting a \mathfrak{C}_{n+1} -cover. Then G has a maximal irredundant core-free $(n+1)$ -cover, $\mathcal{C} = \{M_i \mid i = 1, \dots, n+1\}$ say. Since the Frattini subgroup $\Phi(G)$ of G is contained in M_i for every $i \in \{1, \dots, n+1\}$, $\Phi(G) \leq D_G = 1$, where D is the intersection of the cover \mathcal{C} . Hence $\Phi(G) = 1$ and so G is isomorphic to $(C_p)^{m+1}$ for some positive integer m . Now Propositions 2.2 and 2.5 and Remark 2.3 complete the proof.

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