Tarbiat Moallem University, 20^{th} *Seminar on Algebra,*
2-3 Ordibehesht, 1388 (Apr. 22-23, 2009) pp 166-168

FINITE METACYCLIC p-GROUPS WITH FEW p-AUTOMORPHISMS

REZA ORFI

Department of Mathematics
Arak University
Arak, Iran r-orfi@araku.ac.ir

Abstract. Let G be a non-cyclic p-group of order p^n, $n \geq 3$. It has been conjectured that $|G| \leqslant |\mathrm{Aut}(G)_p|$, where $\mathrm{Aut}(G)_p$ denotes a Sylow p-subgroup of $\mathrm{Aut}(G)$. In this talk we classify the finite non-abelian metacyclic p-groups G, $p > 2$, for which $|G| = |\mathrm{Aut}(G)_p|$. In particular we give an answer to an extended version of a problem posed by Y. Berkovich.

1. Introduction

There have been a number of results on the relationship between the order of a finite p-group G and the order of its automorphism group $\mathrm{Aut}(G)$. It has been conjectured that if G is a non-cyclic p-group of order p^n, $n \geq 3$, then the order of G divides the order of $\mathrm{Aut}(G)$. This has been established for certain classes of finite p-groups. In particular, Davitt in [2] has shown that if G is a non-abelian metacyclic p-group, $p > 2$, then $|G|$ divides $|\mathrm{Aut}(G)|$, in other words $|G| \leq |\mathrm{Aut}(G)_p|$, where $\mathrm{Aut}(G)_p$ denotes a Sylow p-subgroup of $\mathrm{Aut}(G)$. In this talk we classify the finite non-abelian metacyclic p-groups G, $p > 2$, for which $|G| = |\mathrm{Aut}(G)_p|$. This, in particular, solves Problem 2 of [4] for the mentioned class of finite p-groups, giving an answer to an extended version of a problem posed by Y. Berkovich in [1]. Such a classification has already been done by Malinowska for the finite abelian p-groups and finite p-groups of maximal class, see [4].

Throughout this talk the following notation is used. $\mathrm{Hom}(G, A)$ is the group of all homomorphisms of G into an abelian group A; $\mathrm{Aut}_c(G)$ is the group of central automorphisms of G, that is, those automorphisms which induce the identity automorphism on $G/Z(G)$; the split extension of A by B is written as $A \rtimes B$; $|x|$ is the order of x. Also a non-abelian group G that has no non-trivial abelian direct factor is said to be purely non-abelian. By a well-know Theorem (due to Adney) we know that if G is a finite purely non-abelian group then $|\mathrm{Aut}_c(G)| = |\mathrm{Hom}(G/G', Z(G))|$.

2000 Mathematics Subject Classification: 20D15, 20D45.
keywords and phrases: Finite Metacyclic p-Group, Central Automorphism group.

2. Main results

Lemma 2.1. *If G is a finite non-abelian metacyclic p-group, then G is purely non-abelian.*

We below state a Lemma which shows that the metacyclic p-groups G listed in our main Theorem satisfy the relation $|G| = |\mathrm{Aut}(G)_p|$.

Lemma 2.2. *For any positive n, let*

$$G_1 = \langle a, b | a^{p^{n+1}} = b^p = 1, [a, b] = a^{p^n} \rangle,$$
$$G_2 = \langle a, b | a^{p^n} = b^{p^{n+1}} = 1, [a, b] = b^p \rangle \qquad (n > 1).$$

Then $|G_i| = |\mathrm{Aut}(G_i)_p|$, $i = 1, 2$.

Remark . For $G = G_i$ where $i = 1, 2$ we are able to determine the exact order of $\mathrm{Aut}(G)$, by constructing all automorphisms of G. A slightly tedious calculation shows that $\mathrm{Aut}(G) = \mathrm{Aut}(G)_p \rtimes \mathbb{Z}_{p-1}$.

In the rest of this talk we prove the "only if " part of our main Theorem in two cases. First we consider the case $\mathrm{cl}(G) = 2$ and then the case $\mathrm{cl}(G) > 2$.

Lemma 2.3. *Let G be a finite non-abelian metacyclic p-group of class 2. Then G has a presentation of the following form*

$$\langle a, b | a^k = b^{mu}, b^{\ell m} = 1, [a, b] = b^\ell \rangle,$$

where k, m, u and ℓ are some powers of p with $p \le m \le k$, $m \le \ell$ and $1 \le u \le \ell$. Furthermore

(i) *$G' = \langle b^\ell \rangle$, $|b| = \ell m$, $|G| = k\ell m$, and $|a| = k\ell/u$,*
(ii) *$Z(G) \cong \mathbb{Z}_{k\ell/mu} \times \mathbb{Z}_u$ if $k \ge mu$ and $Z(G) \cong \mathbb{Z}_\ell \times \mathbb{Z}_{k/m}$ if $k < mu$,*
(iii) *the order of $\mathrm{Aut}_c(G)$ is obtained as follows:*

$$|\mathrm{Aut}_c(G)| = \begin{cases} k\ell^2 u/m & \text{if } k \ge mu,\ mu \ge \ell \\ k\ell u^2 & \text{if } k \ge mu,\ mu < \ell \\ k^2\ell/m^2 \gcd(k, \ell) & \text{if } k < mu. \end{cases}$$

Through the first case G always denotes a finite non-abelian metacyclic p-group of class 2 with $p > 2$, and the positive integers k, m, u and ℓ satisfy the conditions stated in Lemma 2.3

In what follows we aim to show that if either $Z(G)$ is non-cyclic or $Z(G)$ is cyclic and $m > p$, then $|G| < |\mathrm{Aut}(G)_p|$. We begin with the latter case.

Lemma 2.4. *If $Z(G)$ is cyclic and $m > p$, then $|G| < |\mathrm{Aut}(G)_p|$.*

Lemma 2.5. *If $Z(G)$ is non-cyclic then $|G| < |\mathrm{Aut}(G)_p|$.*

Corollary 2.6. *Let G be a finite non-abelian metacyclic p-group of class 2, $p > 2$. If $|G| = |\mathrm{Aut}(G)_p|$ then $Z(G)$ is cyclic, $|G'| = p$, and*

$$G \cong \langle a, b | a^{p^{n+1}} = b^p = 1, [a, b] = a^{p^n} \rangle.$$

Proposition 2.7. *Let G be a finite non-abelian metacyclic p-group of class 2, $p > 2$. Then $|G| = |\mathrm{Aut}(G)_p|$ if and only if G is isomorphic to the following group:*

$$\langle a, b | a^{p^{n+1}} = b^p = 1, [a, b] = a^{p^n} \rangle.$$

Now in what follows we consider the case $\mathrm{cl}(G) > 2$. The following lemma which plays a substantial role in the case when $\mathrm{cl}(G) > 2$ follows immediately from the observation made by Davitt in [2].

Lemma 2.8. *Let G be a finite non-abelian metacyclic p-group of class greater than 2, where $p > 2$. Then G has a presentation of the following form:*

$$\langle a, b | a^k = b^{\ell s}, b^{\ell m} = 1, [a, b] = b^\ell \rangle,$$

where k, ℓ, s and m are powers of p with $p \leq s \leq m$, $p \leq \ell < m \leq k$. Furthermore

(i) *$G' = \langle b^\ell \rangle$, $|b| = \ell m$, $|G| = k\ell m$, $|a| = km/s$, and $G/G' \cong \mathbb{Z}_k \times \mathbb{Z}_\ell$,*
(ii) *$Z(G) \cong \mathbb{Z}_{k/s} \times \mathbb{Z}_{\ell s/m}$ if $k \geq \ell s$ and $Z(G) \cong \mathbb{Z}_\ell \times \mathbb{Z}_{k/m}$ if $k < \ell s$.*

We note that the parameters k, ℓ, m and s appeared in the above lemma are those used by Davitt in [2]. In order to make use of some results obtained in [2] we shall fix these parameters in the sequel. Also since $\ell < k$ we may choose r such that $k = r\ell$, as chosen in [2].

Lemma 2.9. *With the notation and assumption of Lemma 2.8, if $s \geq r$ then $|\mathrm{Aut}(G)_p| > |G|$ unless $m = k$ and $\ell = p$.*

Lemma 2.10. *With the notation and assumption of Lemma 2.8, if $1 < s < r$ then $|G| < |\mathrm{Aut}(G)_p|$.*

Corollary 2.11. *Let G be a finite non-abelian metacyclic p-group of class greater than 2, $p > 2$. If $|G| = |\mathrm{Aut}(G)_p|$ then*

$$G = \langle a, b | a^{p^n} = b^{p^{n+1}} = 1, [a, b] = b^p \rangle \qquad (n > 1).$$

Proposition 2.12. *Let G be a finite non-abelian metacyclic p-group of class greater than 2 and $p > 2$. Then $|G| = |\mathrm{Aut}(G)_p|$ if and only if G is isomorphic to the following group:*

$$\langle a, b | a^{p^n} = b^{p^{n+1}} = 1, [a, b] = b^p \rangle \qquad (n > 1).$$

Theorem 2.14 *Let G be a finite non-abelian metacyclic p-group, $p > 2$. Then $|G| = |\mathrm{Aut}(G)_p|$ if and only if G is isomorphic to one of the following groups:*

(i) $\langle a, b | a^{p^{n+1}} = b^p = 1, [a, b] = a^{p^n} \rangle$,
(ii) $\langle a, b | a^{p^n} = b^{p^{n+1}} = 1, [a, b] = b^p \rangle \qquad (n > 1)$.

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