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ON CONNECTIONS BETWEEN ROUGH SET THEORY AND MV-ALGEBRAS

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Abstract. In this talk, by considering the notion of MV-algebra, we are concerned with a relationship between rough set and MV-algebra theory. We shall introduce the notion of rough subalgebra (resp. ideal) with respect to an ideal of an MV-algebra, which is an extended notion of subalgebra (resp. ideal) in an MV-algebra. Also we shall give some properties of the lower and the upper approximations in an MV-algebra.

1. Introduction

The theory of rough sets was proposed by Pawlak [4] in 1982. The theory of rough sets is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. Davvaz [3], introduced the notion of rough subrings (respectively ideal) with respect to an ideal of a ring, also see [2]

C. C. Chang in [1] introduced the notion of MV-algebra to provide an algebraic proof of the completeness theorem of infinite valued Lukasiewicz propositional calculus. An MV-algebra A is an abelian monoid $< A, 0, \oplus >$ equipped with an operation $*$ such that $(x^*)^* = x$, $x \oplus 0^* = 0^*$ and finally, $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$. If we define the constant $1 := 0^*$ and the auxiliary operations $\odot$, $\vee$, and $\wedge$ by $a \odot b := (a^* \oplus b^*)^*$, $a \vee b := a \oplus (b \odot a^*)$ and $a \wedge b := a \odot (b \oplus a^*)$, then $(M, \odot, 1)$ is a commutative monoid and the structure $(M, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. Also, we define the binary operation $\ominus$ by $x \ominus y := x \odot y^*$. Now, if we define $x \leq y$ if and only if $x \wedge y = x$ for each $x, y \in M$, then according to [1], $\leq$ is an order relation over M. If the order relation $\leq$ defined over M, is total, then we say that M is linearly ordered. We write nx instead of $x \oplus \cdots \oplus x (n-times)$. Also, we define *the order* of an element x, denoted by $ord(x)$, is the least integer m such that $mx = 1$. If no such integer m exists then we write $ord(x) = \infty$. We say MV-algebra M is *locally finite* if and only if, every element of M different from 0 has a finite order. Let X be a subset of an MV-algebra M. Chang in [1], has shown that every locally finite MV-algebra is linearly ordered. As usual, we say that X is

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an MV-subalgebra (for short, subalgebra) of M if and only if X is closed under the MV-operations defined in M. In an MV-algebra M, the distance function is defined by $d : M \times M \longrightarrow M$, where $d(a,b) := (a^* \odot b) \oplus (b^* \odot a)$. Let M be an MV-algebra and I a nonempty subset of M. Then we say that I is an ideal if the following conditions are satisfied: (1) $0 \in I$, (2) $x, y \in I$ imply $x \oplus y \in I$, and (3) $x \in I$ and $y \leq x$ imply $y \in I$. A proper ideal $P \in I(M)$ is called *prime* whenever $x \wedge y \in P$, then either $x \in P$ or $y \in P$. The set of all prime ideals of an MV-algebra M shall be denoted by $spec(M)$. Let M be an MV-algebra and I is an ideal of M. Then the relation was induced by I, defined as: $x \sim_I y \equiv d(x,y) \in I$ is a congruence relation. The class of equivalence relation of $x \in M$ respected to I is denoted by $[x]_I$. Let M be a linearly ordered MV-algebra and X a subset of M. Then X is called *convex* if for every $x, y \in X$ and $z \in M$, $x \leq z \leq y$ implies $z \in X$.

Proposition 1.1. *Let I be an ideal of a linearly ordered MV-algebra M. Then $[x]_I$ is convex for each $x \in M$.*

A pair (U, θ), where $U \neq \emptyset$ and θ is an equivalence relation on U, is called an approximation space. For an approximation space (U, θ), by a rough approximation in (U, θ) we mean a mapping $Apr : P(U) \longrightarrow P(U) \times P(U)$ defined for every $X \in P(U)$ by $Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$, where $\overline{Apr}(X) = \{x \in U : [x]_\theta \subseteq X\}$, $\underline{Apr}(X) = \{x \in U : [x]_\theta \cap X \neq \emptyset\}$. $\underline{Apr}(X)$, where $[x]_\theta$ is the equivalence class of x, is called a lower rough approximation of X in (U, θ). Also, $\overline{Apr}(X)$ is called upper rough approximation of X in (U, θ). If $\underline{Apr}(X) = \overline{Apr}(X)$, then X is called definable with respect to *theta*. If $\underline{Apr}(X) = \emptyset$, then X is called empty interior respect to θ.

2. Main results

Throughout this paper M is an MV-algebra. Let I be an ideal of M and X be a nonempty subset of M. Then the sets $\underline{Apr}_I(X) = \{x \in M | [x]_I \subseteq M\}$ and $\overline{Apr}_I(X) = \{x \in M | [x]_I \cap M \neq \emptyset\}$ are called, respectively, lower and upper approximations of the set X with respect to the ideal I.

Example 2.1. Let $S_7 = \{0, 1/7, 2/7, \cdots, 6/7, 1\}$. We define $p/7 + q/7 := \min\{(p+q)/7, 1\}$ and $(p/7)^* := (7-p)/7$, then $(S_7, +, *, 0)$ is an MV-algebra. Now, let θ be an equivalence relation with following equivalence classes: $E_1 = \{0, 3/7, 4/7\}$, $E_2 = \{1/7, 6/7\}$, $E_3 = \{2/7\}$, $E_4 = \{5/7\}$. Let $X := \{2/7, 4/7\}$, then $\underline{Apr}(X) = \{2/7\}$ and $\overline{Apr}(X) := \{0, 2/7, 3/7, 4/7\}$.

Proposition 2.2. *Let I be an ideal of M and X a non-empty set of M. Then $\overline{Apr}_I(X)^* = \overline{Apr}_I(X^*)$ and $\underline{Apr}_I(X)^* = \underline{Apr}_I(X^*)$.*

Proposition 2.3. *Let M be a linearly ordered MV-algebra, I an ideal of M and X a convex subset of M. Then $\overline{Apr}_I(X)$ and $\underline{Apr}_I(X)$ are convex subsets.*

Let X be a non-empty subset of an MV-algebra M, and $X^\perp$ be the annihilator of X in M defined by $X^\perp = \{a \in M : a \wedge x = 0, \text{ for all } x \in X\}$. If $X = \{x\}$, then we write $x^\perp$ for $X^\perp$.

Proposition 2.4. *Let I be an ideal of M and X a non-empty set of M. Then $\underline{Apr}_I(X^{\perp}) \subseteq \underline{Apr}_I(X)^{\perp}$, $\overline{Apr}_I(X)^{\perp} \subseteq \overline{Apr}_I(X^{\perp})$ and $\overline{Apr}_I(X)^{\perp} \subseteq \underline{Apr}_I(X)^{\perp}$.*

Example 2.5. Let $M = \{0, x_1, x_2, x_3, x_4, 1\}$. Consider the following tables:

$\oplus$	0	x_1	x_2	x_3	x_4	1
0	0	x_1	x_2	x_3	x_4	1
x_1	x_1	x_3	x_4	x_3	1	1
x_2	x_2	x_4	x_2	1	x_4	1
x_3	x_3	x_3	1	x_3	1	1
x_4	x_4	1	x_4	1	1	1
1	1	1	1	1	1	1

$*$	0	x_1	x_2	x_3	x_4	1
	1	x_4	x_3	x_2	x_1	0

Then $(M, \oplus, *, 0)$ is an MV-algebra. Let $X = \{0, x_2, x_4, 1\}$ and $Y = \{0, x_1, x_3\}$ be subsets of M and $I = \{0, x_2\}$ the ideal of M. It is easy to check that $X^{\perp} = \{0\}$ and $Y^{\perp} = \{0, x_2\}$, so we have $\underline{Apr}_I(X^{\perp}) = \emptyset$, $\underline{Apr}_I(X)^{\perp} = \{0, x_1\}$, $\overline{Apr}_I(X)^{\perp}) = \{0\}$, $\overline{Apr}_I(Y^{\perp}) = \{0, x_2\}$, and $\overline{Apr}_I(Y)^{\perp} = \{0\}$, so $\underline{Apr}_I(X)^{\perp} \nsubseteq \underline{Apr}_I(X^{\perp})$, $\overline{Apr}_I(Y^{\perp}) \nsubseteq \overline{Apr}_I(Y)^{\perp}$ and $\underline{Apr}_I(X)^{\perp} \nsubseteq \overline{Apr}_I(X)^{\perp}$.

Let X and Y be non-empty subsets of M. Then we have

$$X + Y = \{a \in M : a \leq x \oplus y, x \in X, y \in Y\}.$$

If either X or Y are empty, then we define $X + Y = \emptyset$. Clearly, $X + Y = Y + X$ for every $X, Y \subseteq M$. If I and J are two subalgebras or ideals of an MV-algebra M, we can show that $I + J$ is the smallest ideal such that contained I and J. In fact $I + J$ is the ideal generated by $I \cup J$. Moreover, if I, J and K are three ideals of M such that $I \subseteq K$ and $J \subseteq K$ then we obtain $I + J \subseteq K$.

Proposition 2.6. *Let I be an ideal of an MV-algebra of M and X, Y non-empty subsets of M. Then $\overline{Apr}_I(X+Y) \subseteq \overline{Apr}_I(X) + \overline{Apr}_I(Y)$. Particularly, If M is a linearly ordered MV-algebra, then $\overline{Apr}_I(X+Y) = \overline{Apr}_I(X) + \overline{Apr}_I(Y)$.*

Lemma 2.7. *Let I be an ideal of MV-algebra M and X non-empty subset of M. Then X is definable if and only if $\underline{Apr}_I(X) = X$ or $\overline{Apr}_I(X) = X$.*

Proposition 2.8. *Let M be an MV-algebra, I an ideal of M and X, Y subsets of M such that $X + I = X$ or $Y + I = Y$. Then $X + Y$ is a definable set with respect to I. Particularly, If X is an arbitrary subset of M then $X + I$ is a definable set with respect to I.*

Proposition 2.9. *Let I be an ideal of an MV-algebra of M and X, Y non-empty subsets of M. Then $\underline{Apr}_I(X) + \underline{Apr}_I(Y) \subseteq \underline{Apr}_I(X + Y)$.*

Example 2.10. *Let M be a linearly ordered MV-algebra that it is not locally finite and $I \neq 0$ be a proper ideal of M. Let $X = \{0\}$ and $Y = \{1\}$. Clearly, $X+Y = M$ so $\underline{Apr}_I(X+Y) = M$, but one can see that $\underline{Apr}_I(X) + \underline{Apr}_I(Y) = \emptyset$.*

Proposition 2.11. *Let I, J be two ideals of MV-algebra M and X a non-empty subset of M. If $X \subseteq B(M)$ or M is a linearly ordered MV-algebra, then $\overline{Apr}_{I+J}(X) \subseteq \overline{Apr}_I(X) + \overline{Apr}_J(X)$.*

Example 2.12. *Let M be a linearly ordered MV-algebra, $0 = \{0\}$ the ideal of M and $t \neq 0$ an element of M such that $ord(t) \neq 2$. By Proposition 2.8, $t + 0$ is a definable set with respect to ideal 0, so by Proposition 3.13, we have $t + 0 \subseteq t + t + 0$. Now, we claim that $t \oplus t \notin t + 0$. Assume $t \oplus t \in t + 0$, so there exists $s \leq t$ such that $t \oplus t \leq s$. We can obtain that $t = 0$ and it is a contradiction. Hence, it implies that $\overline{Apr}_I(X) + \overline{Apr}_J(X)$ is not a subset of $\overline{Apr}_{I+J}(X)$.*

Proposition 2.13. *Let I, J be two ideals of MV-algebra M and X a non-empty subset of M. Then $\underline{Apr}_{I+J}(X) \subseteq \underline{Apr}_I(X) + \underline{Apr}_J(X)$. Furthermore, if $a \in \underline{Apr}_I(X) + \underline{Apr}_J(X)$ we obtain $[a]_{I+J} \subseteq \underline{Apr}_I(X) + \underline{Apr}_J(X)$. Moreover, If X is an ideal of M, we obtain that $\underline{Apr}_{I+J}(X) = \underline{Apr}_I(X) + \underline{Apr}_J(X)$.*

Proposition 2.14. *Let X be a non-empty subset of M. Then $\bigcap_{P \in spec(M)} \underline{Apr}_P(X) = 0$. Let I, J be two ideals of MV-algebra M and X a non-empty subset of M. If X is an ideal of M and $I, J \subseteq X$, or M is a linearly ordered MV-algebra, then $\underline{Apr}_I(X) \cap \underline{Apr}_J(X) = \underline{Apr}_{I \cap J}(X)$. If X is definable with respect to I or J, or M a linearly ordered MV-algebra then $\overline{Apr}_{I \cap J}(X) = \overline{Apr}_I(X) \cap \overline{Apr}_I(X)$.*

Proposition 2.15. *Let M be an MV-algebra and I an ideal of M. If X is a subalgebra of M, then $\overline{Apr}_I(X)$ is a subalgebra too. In particular, if M is a linearly ordered MV-algebra and J an ideal of M then $\overline{Apr}_I(J)$ is an ideal of M.*

Proposition 2.16. *Let I and J be two ideals of M. Then $\underline{Apr}_I(J)$ is an ideal when $I \subseteq J$ and J is not empty interior. Furthermore, if M is linearly ordered then J is definable or $Apr_I(J) = (0, 0)$.*

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