

## COFINITENESS OF LOCAL COHOMOLOGY MODULES OF DIMENSION ONE IDEALS IN NON-LOCAL CASE

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ABSTRACT. Let  $R$  be a noetherian ring,  $\mathfrak{a}$  an ideal of  $R$  such that  $\dim R/\mathfrak{a} = 1$  and  $M$  a finite  $R$ -module. We will study cofiniteness and some other properties of the local cohomology modules  $H_{\mathfrak{a}}^i(M)$ .

### 1. INTRODUCTION

Throughout  $R$  is a commutative noetherian ring. By a finite module we mean a finitely generated module. For basic facts about commutative algebra see [2] and [8] and for local cohomology we refer to [1].

Grothendieck [6], made the following.

**Conjecture:** For every ideal  $\mathfrak{a}$  and every finite  $R$ -module  $M$ , the module  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^n(M))$  is finite for all  $n$ .

Hartshorne [7] showed that this is false in general. However, he defined  $\mathfrak{a}$ -cofinite modules and he asked the following question.

**Question:** Suppose that  $\mathfrak{a}$  is an ideal of  $R$  and  $M$  is a finite  $R$ -module. When is  $\text{Ext}_R^i(R/\mathfrak{a}, H_{\mathfrak{a}}^j(M))$  finite for every  $i$  and  $j$ ?

Hartshorne [7] showed that if  $(R, \mathfrak{m})$  is a complete regular local ring and  $M$  a finite  $R$ -module, then  $H_{\mathfrak{a}}^i(M)$  is  $\mathfrak{a}$ -cofinite in the following two cases.

- (a)  $\mathfrak{a}$  is a nonzero principal ideal.
- (b)  $\mathfrak{a}$  is a prime ideal with  $\dim R/\mathfrak{a} = 1$ .

Yoshida [9] and Delfino and Marley [3] extended (b) to all dimension one ideals  $\mathfrak{a}$  of an arbitrary local ring  $R$ .

An  $R$ -module  $M$  has *finite Goldie dimension* if  $M$  contains no infinite direct sum of submodules. For a commutative noetherian ring, this can be expressed in two other ways, namely that the injective hull  $E(M)$  of  $M$  decomposes as a finite direct sum of indecomposable injective modules or that  $M$  is an essential extension of a finite submodule.

A module  $M$  is *weakly Laskerian*, whenever for each submodule  $N$  of  $M$ , the quotient  $M/N$  has just finitely many associated primes, see [5]. A module  $M$  is  $\mathfrak{a}$ -*weakly cofinite* if  $\text{Supp}_R(M) \subset V(\mathfrak{a})$  and  $\text{Ext}_R^i(R/\mathfrak{a}, M)$  is weakly Laskerian

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for all  $i$ . Clearly each  $\mathfrak{a}$ -cofinite module is  $\mathfrak{a}$ -weakly cofinite, but the converse is not true in general see [4, Example 3.5 (i) and (ii)].

A module  $M$  is *soclefree* if it has no simple submodules, or in other words  $\text{Ass } M \cap \text{Max } R = \emptyset$ . For example if  $M$  is a module over the local ring  $(R, \mathfrak{m})$  then the module  $M/\Gamma_{\mathfrak{m}}(M)$ , where  $\Gamma_{\mathfrak{m}}(M)$  is the submodule of  $M$  consisting of all elements of  $M$  annihilated by some high power  $\mathfrak{m}^n$  of the maximal ideal  $\mathfrak{m}$ , is always soclefree.

In 2.3, we give a characterization of  $\mathfrak{a}$ -cofiniteness of local cohomology modules with case where  $\mathfrak{a}$  is a one-dimensional ideal in a non-local ring. In this situation we also prove in 2.7, that the local cohomology modules always belong to a class introduced by Zöschinger in [10].

## 2. MAIN RESULTS

**Proposition 2.1.** *Let  $M$  be a module over the noetherian ring  $R$ . The following statements are equivalent.*

- (i)  $M$  is a finite  $R$ -module.
- (ii)  $M_{\mathfrak{m}}$  is a finite  $R_{\mathfrak{m}}$ -module for all  $\mathfrak{m} \in \text{Max } R$  and  $\text{Min}_R(M/N)$  is a finite set for all finite submodules  $N \subset M$ .

**Corollary 2.2.** *Let  $M$  be an  $R$ -module such that  $\text{Supp } M \subset V(\mathfrak{a})$  and  $M_{\mathfrak{m}}$  is  $\mathfrak{a}R_{\mathfrak{m}}$ -cofinite for each maximal ideal  $\mathfrak{m}$ . The following statements are equivalent.*

- (i)  $M$  is  $\mathfrak{a}$ -cofinite.
- (ii) For all  $j$ ,  $\text{Min}_R(\text{Ext}_R^j(R/\mathfrak{a}, M)/T)$  is a finite set for each finite submodule  $T$  of  $\text{Ext}_R^j(R/\mathfrak{a}, M)$ .

**Corollary 2.3.** *Let  $\mathfrak{a}$  an ideal of  $R$  such that  $\dim R/\mathfrak{a} = 1$ ,  $M$  a finite  $R$ -module and  $i \geq 0$ . The following statements are equivalent:*

- (i)  $H_{\mathfrak{a}}^i(M)$  is  $\mathfrak{a}$ -cofinite.
- (ii) For all  $j$ ,  $\text{Min}_R(\text{Ext}_R^j(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M))/T)$  is a finite set for each finite submodule  $T$  of  $\text{Ext}_R^j(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M))$ .

**Corollary 2.4.** *If  $H_{\mathfrak{a}}^i(M)$  (with  $\dim R/\mathfrak{a} = 1$ ) is an  $\mathfrak{a}$ -weakly cofinite module, then it is also  $\mathfrak{a}$ -cofinite.*

Next we will introduce a subcategory of the category of  $R$ -modules that has been studied by Zöschinger in [10, Satz 1.6].

**Theorem 2.5. (Zöschinger)** *For any  $R$ -module  $M$  the following statements are equivalent.*

- (i)  $M$  satisfies the minimal condition for submodules  $N$  such that  $M/N$  is soclefree.
- (ii) For any descending chain  $N_1 \supset N_2 \supset N_3 \supset \dots$  of submodules of  $M$ , there is  $n$  such that the quotients  $N_i/N_{i+1}$  have support in  $\text{Max } R$  for all  $i \geq n$ .
- (iii) With  $L(M) = \bigoplus_{\mathfrak{m} \in \text{Max } R} \Gamma_{\mathfrak{m}}(M)$ , the module  $M/L(M)$  has finite Goldie dimension, and  $\dim R/\mathfrak{p} \leq 1$  for all  $\mathfrak{p} \in \text{Ass}_R(M)$ .

If they are fulfilled, then for each monomorphism  $f : M \longrightarrow M$ ,

$$\text{Supp}_R(\text{Coker } f) \subset \text{Max } R.$$

We will say that  $M$  is in the class  $\mathcal{Z}$  if  $M$  satisfies the equivalent conditions in 2.5.

**Proposition 2.6.** *The class  $\mathcal{Z}$  is a Serre subcategory of the category of  $R$ -modules, that is  $\mathcal{Z}$  is closed under taking submodules, quotients and extensions.*

**Theorem 2.7.** *Let  $N$  be a module over a noetherian ring  $R$  and  $\mathfrak{a}$  an ideal of  $R$  such that  $\dim R/\mathfrak{a} = 1$ . If  $N_{\mathfrak{m}}$  is  $\mathfrak{a}R_{\mathfrak{m}}$ -cofinite for all  $\mathfrak{m} \in \text{Max } R$ , then  $N$  is in the class  $\mathcal{Z}$ . In particular, if  $M$  is a finite  $R$ -module then  $H_{\mathfrak{a}}^i(M)$  is in the class  $\mathcal{Z}$  for all  $i$ .*

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