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ON THE COMMUTATIVITY DEGREE IN ALGEBRAIC STRUCTURES

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Abstract. The commutativity degree of groups and rings has been studied by certain authors since 1973, and the main result obtained is $Pr(A) \leqslant \frac{5}{8}$, where $Pr(A)$ is the commutativity degree of a non-abelian group (or ring) A. Verifying this inequality for an arbitrary semigroup A is a natural question and in this paper by presenting an infinite class of finite non-commutative semigroups we prove that the commutativity degree may be arbitrarily close to 1. We name this class of semigroups the almost commutative or approximately abelian semigroups.

1. Introduction

For a given finite algebraic structure A, the *commutativity degree* of A, denoted by $Pr(A)$, is defined as the probability of choosing a pair (x, y) of the elements of A such that x commutes with y. So,

$$Pr(A) = \frac{| \{(x, y) \in A^2 \mid xy = yx\} |}{| A^2 |} = \frac{\sum_{x \in A} | C_A(x) |}{| A^2 |},$$

where $C_A(x)$ is the centralizer of x in A. For a finite group A it is proved that $Pr(A) = \frac{k(A)}{|A|}$, where $k(A)$ is the number of conjugacy classes of A (see [2, 4, 3, 1, 5], for example). The computational results on $Pr(A)$ are mainly due to Gustafson [2] who shows that $Pr(A) \leqslant \frac{5}{8}$ for a finite non-abelian group A, and MacHale [4] who proves this inequality for a finite non-abelian ring. The groups studied by Lescot [3] mainly satisfy $\frac{1}{2} \leqslant Pr(A) \leqslant \frac{5}{8}$ and the recently obtained results of Doostie [1] concern the groups with the property $d(A) < \frac{1}{2}$, where in that paper $d(A)$ is used instead of $Pr(A)$.

For a finite non-abelian semigroup A, is $Pr(A) \leqslant \frac{5}{8}$? This is a natural question and by considering the presentations

$$\pi_1 = \langle a, b \mid a^m = b^n, aba^l b^k = 1 \rangle,$$

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$$\pi_2 = \langle a, b \mid a^m = b^n, a^2ba^lb^k = a \rangle,$$

and

$$\pi_3 = \langle a, b \mid a^m = b^n, a^2ba^lb^{k+1} = ab \rangle,$$

of the groups, monoids and or, semigroups, where m, n, l and k are any positive integers, we show that $\frac{5}{8}$ is not an upper bound for $Pr(A)$, by providing a class of finite and non-abelian semigroups. Indeed, we show that $Pr(A)$ is arbitrarily close to 1 and we name these kinds of semigroups the *almost commutative* or *approximately abelian* semigroups.

We recall the notion of a presentation $\langle A \mid R\rangle$ of a semigroup. For an alphabet A let A^+ be the free semigroup over A. For a subset R of $A^+ \times A^+$, let ρ be a congruence relation generated by R, then the semigroup $S = A^+/\rho$ will be denoted by $\langle A \mid R\rangle$ which is called a semigroup presentation for S. To lessen the likelihood of confusion, for $\omega_1, \omega_2 \in A^+$ we write $\omega_1 \equiv \omega_2$ if ω_1 and ω_2 are identical words, and $\omega_1 = \omega_2$ if they represent the same element of S (i.e. if $(\omega_1, \omega_2) \in \rho$). Thus, for example, if $A = \{a, b\}$ and R is $\{ab = ba\}$, then $aba = a^2b$ but $aba \not\equiv a^2b$.

To avoid confusion we denote a semigroup presentation by $Sg(\pi)$ and a group presentation by $Gp(\pi)$.

2. Main results

Our main results are:

Theorem 2.1. *Let $S = Sg(\pi_2)$ and $G = Gp(\pi_2)$. If G is abelian then so is S and $Pr(S) = 1$. If G is non-abelian and finite then S is also non-abelian and finite. Moreover,*

$$Pr(S) = \frac{\mid G\mid^2 \cdot Pr(G) + 2(n-1)\cdot \mid \langle b\rangle \mid + (n-1)^2}{\mid S\mid^2}.$$

Theorem 2.2. *Let $S = Sg(\pi_3)$. If S is finite and the minimal two-sided ideal of S is abelian then,*

$$Pr(S) = \frac{m^2 + n^2 + 4mn - mn(m+n) - 3(m+n) + 2 + \mid S\mid^2}{\mid S\mid^2}.$$

Moreover, the semigroup S is never abelian and for all positive integers m, n, l and k if $m \mid l$ and $n \mid k$, then S is finite and $Pr(S) > \frac{5}{8}$.

As a corollary of Theorem 2.2 we conclude that, for all fixed values of m and n,

$$\lim_{l,\, k\to\infty} Pr(S) = 1.$$

For this reason we call the family of semigroups $S = Sg(\pi_3)$ (where the minimal two-sided ideal is abelian) *almost commutative* or *approximately abelian.*

References

[1] H. Doostie, M. Maghasedi, Certain classes of groups with commutativity degree $d(G) < \frac{1}{2}$, *Ars Combinatoria* **89** (2008), 263-270.

[2] W.H. Gustafson, What is the probability that two group elements commute?, *Amer. Math. Monthly* **80** (1973) 1031-1034.

[3] P. Lescot, Isoclinism classes and comutativity degree of finite groups, *J. Algebra* **177** (1995) 847-869.
[4] D. MacHale, Commutativity in finite rings, *Amer. Math. Monthly* **83** (1976) 30-32.
[5] G. Sherman, What is the probability an automorphism fixes a group element?, *Amer. Math. Monthly* **82** (1975) 261-264.