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THE PROPERTIES OF n- SINGER GENERATOR AND NON-COMMUTING SUBSETS OF FINITE THREE-DIMENSIONAL GENERAL LINEAR GROUPS

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Abstract. Let G be a group. A subset N of G is said to be non-commuting if $xy \neq yx$ for any two distinct elements x and y in N. If $|N| \geq |M|$ for any other non-commuting subset M in G, then N is said to be a maximal non-commuting subset. In this paper we obtain lower bounds for the cardinality of a maximal non-commuting subset, by n-Singer generator, in a three-dimensional general linear group. Moreover we obtain structural information about every maximal non-commuting subset containing no proper powers.

1. Introduction

Let G be a non-abelian group and $Z(G)$ be its center. We call a subset N of G *non-commuting* if $xy \neq yx$ for any distinct elements x, y in N. If $|N| \geq |M|$ for any other non-commuting subset M in G, then N is said to be a *maximal non-commuting subset*. The cardinality of such a subset is denoted by $\omega(G)$. By a famous result of Neumann [4] answering a question of P. Erdös, we know that the finiteness of $\omega(G)$ in G implies the finiteness of the factor group $\frac{G}{Z(G)}$.

For a prime number p, a finite p-group G is called extra-special if the center, the Frattini subgroup and the derived subgroup of G all coincide and are cyclic of order p. The cardinalities of maximal non-commuting subsets of extra-special p-groups are important as they provide combinatorial information which can be used to calculate their cohomology lengths. (The cohomology length of a non-elementary abelian p-group is a cohomology invariant defined as a result of a theorem of Serre [6]). Y. M. Chin [3] has obtained upper and lower bounds for the cardinality of maximal non-commuting subsets of extra-special p-groups, for odd prime numbers p. For $p = 2$, it has been shown by Isaacs (see[2, p. 40]) that $\omega(G) = 2n + 1$ for any extra-special group of order 2^{2n+1}. Also in [1, Lemma 4.4], it was proved that $\omega(GL(2,\ q)) = q^2 + q + 1$. In this paper we consider maximal non-commuting subsets in general linear groups of dimension three over a finite field of order q, and obtain lower bounds for their cardinality.

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2. Main result

Theorem 2.1. *Let $G = GL(3,\ q)$. Then $\omega(G) \geq q^6$ if $q \geq 3$, and if $q = 2$ then $\omega(G) = 57$.*

3. n-Singer generator

Definition 3.1. Let $g \in GL(n,\ q)$ where $q = p^k$, p a prime, and $|g| = q^n - 1$. Then $\langle g \rangle$ is called a *Singer cycle subgroup* of G.

Definition 3.2. Let V be a vector space over a finite field F with dimension n. We call $V = V_{n_1} \oplus V_{n_2} \oplus \dots V_{n_k}$ an $(n_1, n_2, \dots, n_k)$-*decomposition* if $(n_1, n_2, \dots, n_k)$ is a partition of n and for $i = 1, 2, \dots, k$, V_{n_i} is a subspace of V of dimension n_i.

Definition 3.3. Let V be an n-dimensional vector space over a finite field F and $V = V_{n_1} \oplus V_{n_2} \oplus \dots \oplus V_{n_k}$ be an $(n_1, n_2, \dots, n_k)$-decomposition of V. An element g of G is called an $(n_1, n_2, \dots, n_k)$-*Singer generator* if $g = g_{n_1} g_{n_2} \dots g_{n_k}$ where, for each i, $\langle g_{n_i} \rangle$ is a Singer cycle subgroup of $GL(V_{n_i})$, and if $n_i = n_j$ then $c_{g_{n_i}}(t) \neq c_{g_{n_j}}(t)$ for all $i \neq j$, where $c_{g_{n_i}}(t)$ is the characteristic polynomial for g_{n_i} on V_{n_i}. We call $\Pi_{i=1}^{k} \langle g_{n_i} \rangle$ the $(n_1, n_2, \dots, n_k)$-*maximal torus corresponding to g*.

Note it is not necessary that a group has an $(n_1, n_2, \dots, n_k)$-Singer generator for all $(n_1, n_2, \dots, n_k)$ such that $\Sigma_{i=1}^{k} n_i = n$. For example existence of a $(1, 1, 1)$-Singer generator element, $g_1 g_2 g_3$ in $GL(3, 2)$ requires three distinct linear polynomials $c_{g_{n_i}}(t) = t - \lambda_i$ and hence requires $q \geq 4$.

Lemma 3.4. *Let $G = GL(3, q)$, where $q = p^k \geq 4$. Suppose that $g \in G$ is an $(n_1, \dots, n_k)$-Singer generator, where $(n_1, \dots, n_k)$ is a partition of 3. Then $C_G(g) = \Pi_{i=1}^{k} \langle g_{n_i} \rangle$.*

4. Proof of Theorem 1.1

Definition 4.1. Let N be a maximal non-commuting subset of finite group G. An element $x \in N$ is called a *proper power* if there exists y in G such that $x = y^k$ and $|x| < |y|$.

Lemma 4.2. *Let G be a group and N a maximal non-commuting subset of G. Suppose $x \in N$ and there exists y such that $x = y^s$ and $|x| < |y|$. Then $\bar{N} = (N \backslash \{x\}) \cup \{y\}$ is a non-commuting subset of G. So G has a maximal non-commuting subset that contains no proper powers.*

Lemma 4.3. *Let $G = GL(3,\ q)$, where $q = p^k > 2$, and let N_3 consist of one (3)-Singer generator of G corresponding to each (3)-maximal torus of G. Then N_3 is a non-commuting subset of size $\frac{|G|}{(q^3-1)3}$. Moreover, if N is a maximal non-commuting subset containing no proper powers, then N contains (3)-Singer generator of each (3)-maximal torus.*

Lemma 4.4. *Let $G = GL(3,\ q)$, where $q = p^k > 2$. Let N_{12} consist of one $(1, 2)$-Singer generator element of G corresponding to each $(1, 2)$-maximal torus of G. Then N_{12} is a non-commuting subset of size $\frac{q^3(q^3-1)}{2}$. Moreover, if N is a maximal non-commuting subset containing no proper powers, then N contains a $(1, 2)$-Singer generator of each $(1, 2)$-maximal torus.*

Lemma 4.5. *Let $G = GL(3,\ q)$, where $q = p^k \geq 4$. Let N_{111} consist of one $(1,1,1)$-Singer generator element of G corresponding to each $(1,1,1)$-maximal torus of G. Then N_{111} is a non-commuting subset of size $\frac{|G|}{6(q-1)^3}$. Moreover, if N is a maximal non-commuting subset containing no proper powers, then N contains a $(1,1,1)$-Singer generator of each $(1,1,1)$-maximal torus.*

Lemma 4.6. *Let $G = GL(3,\ q)$, where $q = p^k \geq 4$. Let $x, y, z \in G$ be a (3)-Singer generator, $(1,2)$-Singer generator and $(1,1,1)$-Singer generator element, respectively. Then $xy \neq yx$, $xz \neq zx$ and $yz \neq zy$.*

Proof of Theorem 1.1

By Lemmas 4.3, 4.4, 4.5 and 4.6, $N_3 \cup N_{12} \cup N_{111}$ is a non-commuting subset of size $\frac{|G|}{3(q^3-1)} + \frac{q^3(q^3-1)}{2} + \frac{|G|}{6(q-1)^3} = q^6$.

Moreover we have shown that every maximal non-commuting subset N of G that contains no proper power, must contains a subset of the form $N_3 \cup N_{12} \cup N_{111}$.

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