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SOME RESULTS ON PSEUDO BCK-ALGEBRAS

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Abstract. Some equivalent conditions for a pseudo BCK-algebra to be commutative is investigated and that when a bounded commutative pseudo BCK-algebra can be a lattice is verified. Finally, it is shown that any pseudo BCK-algebra, under suitable conditions, is a Heyting algebra.

1. Introduction and Preliminaries

In [3], Y. Imai and K. Iséki introduced a new class of algebras called BCK-algebras, which are a generalization of set-theoretic difference and propositional calculus. In virtue of this fact that the operation of set-difference is not commutative, G. Georgescu et al., defined a generalization of BCK-algebras called pseudo BCK-algebras as a structure $\mathfrak{X} = (X, \leq, *, \diamond, 0)$ in which X is a non-empty set, $\leq$ is a binary relation in X, $*$ and $\diamond$ are two binary operations on X and 0 is a fixed element of X satisfying the axioms: $\forall x, y, z \in X$,

(PsBCK1) $(x * y) \diamond (x * z) \leq z * y$, $(x \diamond y) * (x \diamond z) \leq z \diamond y$,
(PsBCK2) $x * (x \diamond y) \leq y$, $x \diamond (x * y) \leq y$,
(PsBCK3) $x \leq x$,
(PsBCK4) $0 \leq x$,
(PsBCK5) $x \leq y \leq x \Rightarrow x = y$,
(PsBCK6) $x \leq y \Leftrightarrow x * y = 0 \Leftrightarrow x \diamond y = 0$.

Clearly, if in a pseudo BCK-algebra the equality $x * y = x \diamond y$ holds, it is a BCK-algebra (see [2]).

J. Kühr [4], introduced commutative pseudo BCK-algebras and investigated some related results. We give a dual form of J. Kühr's definition:

A pseudo BCK-algebra $(X, \leq, *, \diamond, 0)$ is said to be *commutative* if verifies the axioms

$$x * (x \diamond y) = y * (y \diamond x),\ x \diamond (x * y) = y \diamond (y * x).$$

2. Main results

Proposition 2.1. *In a pseudo BCK-algebra the following hold:*

$$x * (x \diamond (x * y)) = x * y,\ x \diamond (x * (x \diamond y)) = x \diamond y.$$

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Theorem 2.2. *For every pseudo BCK-algebra $\mathfrak{X}$, the following are equivalent:*

(1) $x \leq z$ *and* $z * y \leq z * x$, $z \diamond y \leq z \diamond x$ *imply* $x \leq y$.
(2) $x, y \leq z$ *and* $z * y \leq z * x$, $z \diamond y \leq z \diamond x$ *imply* $x \leq y$.
(3) $x \leq y$ *implies* $y \diamond (y * x) = x = y * (y \diamond x)$.
(4) $y \diamond (y * x) \leq x \diamond (x * y)$, $y * (y \diamond x) \leq x * (x \diamond y)$.
(5) $\mathfrak{X}$ *is commutative.*
for all $x, y, z \in X$.

Theorem 2.3. *In any pseudo BCK-algebra the following are equivalent:*

(1) $x \leq z$ *and* $z * y \leq z * x$ *(or* $z \diamond y \leq z \diamond x$*) imply* $x \leq y$.
(2) $x * y = 0$ *(or* $x \diamond y = 0$*) implies that* $x * (y \diamond (y * x)) = 0 = x \diamond (y * (y \diamond x))$.
for all $x, y, z \in X$,

G. Georgescu et al. [2] proved that any commutative pseudo BCK-algebra with its order is a lower semilattice in which the infimum of two element x and y is given by

$$x \wedge y = x * (x \diamond y) = x \diamond (x * y). \tag{2.1}$$

Now, a condition under which a commutative pseudo BCK-algebra is a lattice is given.

First, we note that a pseudo BCK-algebra $\mathfrak{X}$ is said to be *bounded* if there exists an element $1 \in X$ such that $x \leq 1$, for all $x \in X$. Also, note that in a bounded pseudo BCK-algebra $\mathfrak{X}$, $x * 1 = x \diamond 1 = 0$ while neither $1 * x$ $(1 \diamond x)$ is equal zero unless $x = 1$, nor they are equal in general. So, we use x^* $(x^\diamond)$ to denote $1 * x$ $(1 \diamond x)$, and also $x^{*\diamond}$ for $(x^*)^\diamond$.

Theorem 2.4. *In a bounded pseudo BCK-algebra the following hold:*

$$1 * (1 \diamond x) = x = 1 \diamond (1 * x), \text{ i.e., } x^{*\diamond} = x = x^{\diamond *}.$$

Theorem 2.5. *If* $\mathfrak{X} = (X, \leq, *, \diamond, 0, 1)$ *is a bounded commutative pseudo BCK-algebra, then* $(X, \leq)$ *is a lattice, in which the infimum is as (2.1) and the supremum is given by* $x \vee y = (x^\diamond \wedge y^\diamond)^* = (x^* \wedge y^*)^\diamond$.

In [1], Georgescu et al. introduced the concept of a pseudo MV-algebra as a non-commutative generalization of MV-algebras as an algebra $(M, \oplus, ^-, ^\sim, 0, 1)$ satisfying the following conditions: for all $x, y, z \in M$,

(1) $\oplus$ is associative,
(2) $x \oplus 0 = 0 \oplus x = x$,
(3) $x \oplus 1 = 1 \oplus x = 1$,
(4) $1^\sim = 0$, $1^- = 0$,
(5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,
(6) $(x^-)^\sim = x$,
(7) $x \cdot (x^- \oplus y) = (x \oplus y^\sim) \cdot y$,
(8) $x \cdot y^- \oplus y = y \cdot x^- \oplus x$,
(9) $x \oplus x^\sim \cdot y = y \oplus y^\sim \cdot x = x \cdot y^- \oplus y$

where $x \cdot y = (y^- \oplus x^-)^\sim$.

Let $(M, \oplus, ^-, ^\sim, 0, 1)$ be a pseudo MV-algebra and define

$$x * y = (y \oplus x^\sim)^- \quad \text{and} \quad x \diamond y = (x^- \oplus y)^\sim.$$

Then $(M, *, \diamond, 0, 1)$ is a bounded commutative pseudo BCK-algebra. Conversely, if $(M, *, \diamond, 0, 1)$ is a bounded commutative pseudo BCK-algebra and we define

$$x \oplus y = (x^* * y)^\diamond = (y^\diamond \diamond x)^*, \; x^- = x^\diamond, \; x^\sim = x^*$$

then $(M, \oplus, ^-, ^\sim, 0, 1)$ is a pseudo MV-aglebra.

Moreover, every interval $[0, a]$ of any commutative pseudo BCK-algebra is a bounded commutative pseudo BCK-algebra, hence a pseudo MV-algebra. For more details refer to [5].

Definition 2.6. (i) An algebra $(L, \wedge, \vee, \rightarrow, 1)$ where $(L, \wedge, \vee, 1)$ is a lattice with the greatest element and the binary operation $\rightarrow$ verifies

$$x \leq y \rightarrow z \Leftrightarrow x \wedge y \leq z, \ \forall x, y, z \in L$$

is called *relatively pseudocomplemented*.

(ii) A bounded relatively pseudocomplemented lattice is called a *Heyting algebra*.

Theorem 2.7. *A complete lattice is a Heyting algebra iff it satisfies the identity*

$$x \wedge \bigvee_{\alpha \in \Lambda} y_\alpha = \bigvee_{\alpha \in \Lambda} (x \wedge y_\alpha).$$

Theorem 2.8. *Suppose that* $\mathfrak{X} = (X, *, \diamond, 0, 1)$ *is a bounded commutative pseudo* BCK*-algebra, and is such that it is a complete lattice with respect to the order "*$\leq$*". Then* $(X, \vee, \wedge, ^\star, 0, 1)$ *is a Heyting algebra.*

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