*Tarbiat Moallem University, $20^{th}$ Seminar on Algebra,*
*2-3 Ordibehesht, 1388 (Apr. 22-23, 2009) pp 62-65*

# FUZZY PRIMARY SUBACTS

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Abstract. In this paper the concepts of a fuzzy primary subact is given and some fundamental results are proved. Also a characterization of a fuzzy $P$-primary subact is given.

## 1. Introduction

In [3] the notion of $L$-fuzzy primary ($P$-primary) submodule of $M$ is given in terms of fuzzy singletons. In this paper, we generalize this definition to fuzzy primary ($P$-primary) subact.

Throughout this paper $S$ will denote a commutative monoid with 0. Recall that a *right $S$-act* is a set $A$ together with a function $\lambda : A \times S \to A$, called the *action* of $S$ (or the $S$-action) on $A$, such that for $a \in A$ and $s, t \in S$ (denoting $\lambda(a, s)$ by $as$), $a(st) = (as)t$ and $a1 = a$.

Throughout this paper all acts is *centered right $S$-act* (if there is a unique fixed element in $A$ denoted $\theta$ such that $\theta s = \theta$ and $a0 = \theta$, $\forall s \in S$ and $a \in A$) and $A$ will always denote a centered right $S$-act. We denote by $F(X)$ the set of all fuzzy subsets of $X$. For $\mu, \lambda \in F(X)$, we say $\mu \subseteq \lambda$ iff $\mu(x) \leq \lambda(x)$ for all $x \in X$. Let $\mu \in F(X)$ and $t \in [0, 1]$. Then the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called the level subset of $X$ with respect to $\mu$. Also we put $\mu_* = \{x \in X : \mu(x) \geq 1\}$. $x^r \in F(X)$ is called a *fuzzy point* iff $x^r(y) = 0$ for $y \neq x$, and $x^r(x) = r \neq 0$. The fuzzy point $x^r$ is said to belong to $\mu \in F(X)$, written $x^r \in \mu$, iff $\mu(x) \geq t$. Let $\mu, \lambda \in F(X)$. Then $\mu \subseteq \lambda$ iff $x^t \in \mu$ implies $x^t \in \lambda$ for all fuzzy point $x^t \in FP(X)$. $\lambda \in F(S)$ is called a *fuzzy right (left) ideal* of $S$ if $\lambda \circ \chi_S \subseteq \lambda$ ($\chi_S \circ \lambda \subseteq \lambda$). It is clear that $\mu$ is a fuzzy ideal of $S$ iff $\mu(xy) \geq \mu(x) \vee \mu(y)$. We denote by $FI(S)$, the set of all fuzzy ideal of $S$. Throughout this paper we suppose that if $\lambda$ is a fuzzy ideal of $S$, then $\lambda(0) = 1$.

Let $I$ be an ideal of $S$. Then $I$ is called a *prime ideal* of $S$ if for all $a, b \in S$, $ab \subseteq I$ implies that $a \in I$ or $b \in I$. Note for ideal $I$ of $S$, the notation $\sqrt{I}$ is the intersection of all prime ideal of $S$ containing $I$ and $\sqrt{I} = S$ if $I$ is not contained in any prime ideal of $S$. In [2], it is proved that if $I$ is an ideal $I$ of $S$, then $\sqrt{I} = \{s \in S : s^n \in I \text{ for some } n \in \mathbb{N}\}$. Let $I$ be an ideal of $S$. Then

**2000 Mathematics Subject Classification:** 13D45, 13D07, 20M30 .
**keywords and phrases:** Fuzzy subact, Fuzzy Primary subact and Fuzzy P-Primary subact.

$I$ is called a *primary ideal* of $S$ if for all $a, b \in S$, $ab \subseteq I$ implies that $a \in I$ or $b \in \sqrt{I}$.

Let $\mu \in FI(S)$ be non-constant. Then $\mu$ is called be a *fuzzy prime ideal* of $S$ if for any $\lambda, \gamma \in FI(S)$, $\lambda \circ \gamma \subseteq \mu$ implies that either $\lambda \subseteq \mu$ or $\gamma \subseteq \mu$.

**Theorem 1.1.** [4] *$\mu \in FI(S)$ is a fuzzy prime ideal of $S$ iff for any two fuzzy points $x^r, y^s \in FP(S)$, $x^r \circ y^s \subseteq \mu$ implies that either $x^r \in \mu$ or $y^s \in \mu$.*

Let $\mu \in FI(S)$ be non-constant. Then $\mu$ is called a *fuzzy primary ideal* of $S$ iff for any fuzzy points $x^r, y^s \in FP(S)$, $x^r \circ y^s \subseteq \mu$ implies that either $x^r \in \mu$ or $(y^s)^n \in \mu$ for some $n \in \mathbb{N}$.

We observe that any fuzzy prime ideal is a fuzzy primary ideal.

If $\mu \in FI(S)$, then we put

$$\sqrt{\mu} = \begin{cases} \bigcap\limits_{\mu \subseteq P} P & if\ there\ is\ a\ fuzzy\ prime\ ideal\ P\ such\ that\ \mu \subseteq P \\ \chi_S & otherwise. \end{cases}$$

In [2] a proper subact $B$ of $A$ is said to be *primary subact* of $A$ if for every $s \in S$ and $a \in A$ , $as \in B$ implies that $a \in B$ or $s \in \sqrt{(B : A)}$. Note that if $s \in \sqrt{(B : A)}$, then there is an $n \in \mathbb{N}$ such that $s^n \in (B : A)$. It is not difficult to see that if $B$ is a primary subact of $A$ and $P = \sqrt{(B : A)})$, then $P$ is a prime ideal of S. Thus if $B$ is a primary subact of $A$ and $P = \sqrt{(B : A)}$, we will say that $B$ is a $P$-primary subact. In [2] it is proved that if $I$ is a proper ideal of $S$, then $\sqrt{I} = \{t \in S : \exists n \in \mathbb{N}(t^n \in I)\}$.

## 2. Fuzzy subact

We now define a fuzzy $S$-act of $A$. Let $A$ be a $S$-act and $\mu \in F(A)$. Then $\mu$ is called a *fuzzy right(left) $S$-act* of $A$ if $\mu(as) \geq \mu(a)$ $(\mu(sa) \geq \mu(a))$ for all $a \in A$ and for all $s \in S$ and $\mu(\theta) = 1$. We denote by $FS(A)$ the set of all fuzzy right $S$-act of $A$.

Let $\mu \in F(A)$ and $\lambda \in F(S)$. Define the composition of $\mu$ and $\lambda$ as follows: $(\mu \circ \lambda)(x) = \vee\{\mu(a) \wedge \lambda(s) \mid x = as$ for some $a \in A$ and $s \in S\}$ for all $x \in A$.

**Proposition 2.1.** *If $\mu \in FS(A)$ and $\lambda \in FI(S)$, then $\mu \circ \lambda \in FS(A)$.*

**Lemma 2.2.** *If $A = S$, then $\mu$ is a fuzzy $S$-act of $A$ iff $\mu$ is a fuzzy ideal of $S$.*

**Theorem 2.3.** *Let $\mu \in F(A)$ and $\mu(\theta) = 1$. Then $\mu \in FS(A)$ iff $\mu_t$ is a subact of $A$ for all $t \in Im(\mu)$.*

## 3. Fuzzy Primary subact

In this section we give some characterizations for fuzzy primary subacts of $A$. We start with some definitions of ordinary subact which will be expressed later in fuzzy context.

Let $\mu$, $\lambda \in FS(A)$ be non-constant. Then $\lambda$ is called a *fuzzy subact* of $\mu$ on $A$ iff $\lambda \subset \mu$. If $\mu = \chi_A$ and $\lambda$ be non-constant, then $\lambda$ is called a fuzzy subact of $A$.

Let $\lambda$ be a fuzzy subact of $\mu$. Then $\lambda$ is said to be a *fuzzy primary subact* of $\mu$ on $A$ iff for any fuzzy points $s^\alpha \in FP(S)$ and $a^\beta \in FP(A)$, $a^\beta \circ s^\alpha \subseteq \lambda$ implies that $a^\beta \in \lambda$ or $\mu \circ (s^\alpha)^n \subseteq \lambda$ for some $n \in \mathbb{N}$.

**Proposition 3.1.** *Let $\mu \in F(S)$ and $A = S$. Then $\mu$ is a fuzzy primary subact of $A$ iff $\mu$ is a fuzzy primary ideal of $S$.*

**Proposition 3.2.** *Let $\lambda$ be a fuzzy primary subact of $\mu$ on $A$ and $t \in Im(\lambda)$. If $\lambda_t \neq \mu_t$, then $\lambda_t$ is a primary subact of $\mu_t$.*

**Example 3.3.** Let $A = S = \mathbb{Z}$. We define:

$$\mu(x) = \begin{cases} 1 & if\ x \in 4\mathbb{Z} \\ \frac{1}{2} & if\ x \in 2\mathbb{Z} \setminus 4\mathbb{Z} \\ 0 & otherwise \end{cases} \quad , \quad \lambda(x) = \begin{cases} 1 & if\ x = 0 \\ \frac{1}{2} & if\ x \in 4\mathbb{Z} \setminus \{0\} \\ 0 & otherwise. \end{cases}$$

Then $\lambda_t$ is a primary subact of $\mu_t$ for each $t \in (0, 1]$, but $\lambda$ is not a fuzzy primary subact of $\mu$, since if we put $r = \frac{2}{3}$, $t = \frac{1}{3}$, $y = 4$ and $x = 5$, then $x^t \circ y^r \in \lambda$ but $x^t \notin \lambda$ and $\mu \circ (y^s)^k \nsubseteq \lambda$ for all $k \in \mathbb{N}$.

**Proposition 3.4.** *Let $B$ be a primary subact of $A$ and $\alpha \in [0, 1)$. If $\lambda$ is the fuzzy subset of $A$ defined by $\lambda(x) = 1$ if $x \in B$ and $\lambda(x) = \alpha$ otherwise, for all $x \in A$. Then $\lambda$ is a fuzzy primary subact of $A$.*

**Proposition 3.5.** *If $\mu$ is a fuzzy primary subact of $A$, then there exists $\alpha \in [0, 1)$ and a primary subact $B$ of $A$ such that $\mu(x) = 1$ for $x \in B$ and otherwise $\mu(x) = \alpha$.*

Let $\mu$ be a non-constant ideal in a commutative ring $R$. In 1990 Malik and Mordeson in "*Fuzzy prime ideals of a ring*" have proved that $\mu$ is prime iff $|Im(\mu)| = 2$ and $\mu_*$ be a prime ideal of $R$. The following theorem is a consequence of Propositions 3.4 and 3.5.

**Theorem 3.6.** *$\mu$ is a fuzzy prime ideal of $S$ iff there exists $\alpha \in [0, 1)$ such that $\mu(x) = 1$ if $x \in \mu_*$ and $\mu(x) = \alpha$ if $x \notin \mu_*$, and $\mu_*$ is a prime ideal of $S$.*

## 4. Fuzzy P-Primary S-act

In this section first we describe fuzzy ideal $(\mu : \lambda)$ of $S$, next characterize fuzzy $P$-primary subacts of $A$.

Let $\lambda, \mu \in FS(A)$. Then $(\mu : \lambda)$ is the fuzzy subset of $S$ defined by $(\mu : \lambda)(s) = \vee \{\alpha \in [0, 1] \mid \lambda \circ s^\alpha \subseteq \mu\}$ for all $s \in S$.

**Proposition 4.1.** *If $\lambda, \mu \in FS(A)$, then $(\mu : \lambda)$ is a fuzzy ideal of $S$.*

**Lemma 4.2.** *Let $\lambda, \mu, \omega \in FS(A)$ and $s^\alpha \in FP(S)$. Then*

(1) $\mu \circ (\lambda : \omega) \subseteq \lambda$.
(2) *If $\mu \subseteq \lambda$, then $(\mu : \omega) \subseteq (\lambda : \omega)$ and $(\omega : \lambda) \subseteq (\omega : \mu)$.*
(3) *If $\mu \subseteq \lambda$, then $(\lambda : \mu) = \chi_S$.*

**Proposition 4.3.** *If $\mu \in FS(A)$, then $(\mu : \chi_A)_* = (\mu_* : A)$.*

**Theorem 4.4.** *Let $\lambda$ be a fuzzy primary subact of $A$. Then $\sqrt{(\lambda : \chi_A)}$ is a fuzzy ideal of $S$.*

Let $\mu$ be a fuzzy primary subact of $A$ and $P = \sqrt{(\mu : \chi_A)}$. Then $\mu$ is called a *fuzzy $P$-primary subact* of $A$.

**Proposition 4.5.** *Let $\mu$ be a fuzzy $P$-primary subact of $A$, $a^\beta \in FP(A)$ and $s^\alpha \in FP(S)$. If $a^\beta \circ s^\alpha \in \mu$, then $a^\beta \in \mu$ or $s^\alpha \in P$.*

**Proposition 4.6.** *Let $\mu$ be a fuzzy $P$-primary subact of $A$, $\lambda \in FI(S)$ and $\omega$ be a fuzzy subact of $A$. If $\omega \circ \lambda \subseteq \mu$, then $\omega \subseteq \mu$ or $\lambda \subseteq P$.*

**Theorem 4.7.** *Let $\mu$ be a non-constant fuzzy subact of $A$ and $P \in FI(S)$. Then $\mu$ is a fuzzy $P$-primary subact of $A$ iff*

(1) *If $a^\beta \circ s^\alpha \in \mu$ and $a^\beta \notin \lambda$, then $s^\alpha \in P$ for all fuzzy points $a^\beta \in FP(A)$ and $s^\alpha \in FP(S)$ and*

(2) *If $s^\alpha \in P$, then $\exists n \in \mathbb{N}$ such that $\chi_A \circ (s^\alpha)^n \subseteq \mu$.*

**Theorem 4.8.** *Let $\mu$ be a fuzzy $P$-Primary subact of $A$ and $\lambda \in FS(A)$. If $(\mu : \lambda)$ is non-constant, then $(\mu : \lambda)$ is a fuzzy $P$-Primary ideal of $A$.*

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