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THE PRESERVATION OF EXACT SEQUENCES OF MODULES BY RADICAL AND ZARISKI FUNCTORS

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Abstract. Throughout all rings are commutative with identity and all modules are unitary. Depending on an R-module M, $RAD(M)$ is an R-lattice of radical submodules of M and $\zeta(M)$ is a Zariski space over the semiring $\zeta(R)$. There are two isomorphic full subcategories RAD and ZAR, whose objects are the semimodules $RAD(M)$ and $\zeta(M)$ over the simiring $Id(R)$ of the ideals of R. These provide two naturally isomorphic functors $\mathcal{R}$ and $\mathcal{Z}$ from MOD to $SEMOD$ respectively. Beside the investigation preserving and reflecting exact sequences by $\mathcal{R}$ and $\mathcal{Z}$, It is proved that $\mathcal{R}$ and $\mathcal{Z}$ preserve finite free representations of modules over a domain R.

1. Introduction

A proper submodule P of an R-module M is said to be prime if $rm \in P$ for $r \in R$ and $m \in M$ implies that either $m \in P$ or $rM \subseteq P$. The radical of a submodule N, denoted $radN$, is defined to be the intersection of all prime submodules containing N. A submodule N of an R-module M is said to be a radical submodule, if $radN = N$. Also M is called a radical module or semi-prime, if the zero submodule of M is a radical submodule. The set of all radical submodules of an R- module M , denoted, $RAD(M)$, is an R- lattice with the operations $L \vee N = rad(L + N)$, $L \wedge N = L \cap N$, $I.L = rad(IL)$(cf. [9]). Also if $Id(R)$ is the semiring of ideals of R together with ordinary addition and multiplication of ideals, then $(RAD(M), \vee, .)$ is an $Id(R)$- semimodule. They are considered as objects of the full subcategory RAD of $SEMOD$ the category of $Id(R)$-semimodules. Recall that the variety $V(N)$ of a submodule N of M is defined to be the set of all prime submodues P of M containing N. In general the collection of varieties, denoted $\zeta(M)$, dose not form a topology on $spec(M)$(the set of all prime submodules of M), in spite of what has been seen in rings(cf. [6]). $\zeta(R)$ is a semiring where "addition" is given by intersection and "multiplication" is given by union. Moreover $\zeta(M)$ is a semimodule over $\zeta(R)$ with the addition and scalar multiplication $V(L) + V(N) = V(L) \cap V(N) = V(L + N)$, $V(I) * V(N) = V(IN)$. It induces an $Id(R)$- semimodule structure for $\zeta(M)$

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with the same addition and scalar multiplication as $I.V(N) = V(IN)$. (For further reading about semirings, semimodules, and Zariski spaces, see for example [2], [8], or [9]). ZAR is an another full subcategory of $SEMOD$ whose objects are the Zariski spaces $\zeta(M)$. These lead us to define a natural transformation τ between radical functor $\mathcal{R}$ and Zariski functor $\mathcal{Z}$ which are given as follows; $\mathcal{R} : MOD \longrightarrow SEMOD$ defined on the R-module M by $\mathcal{R}(M) = RAD(M)$ and on the module homomorphism $f : M \rightarrow M^{'}$ by $\mathcal{R}(f)(L) = rad(f(L))$ and $\mathcal{Z} : MOD \longrightarrow SEMOD$ defined on the R-module M by $\mathcal{Z}(M) = \zeta(M)$ and on the module homomorphism $f : M \rightarrow M^{'}$ by $\mathcal{Z}(f)(V(L)) = V(f(L))$. In algebraic view McCasland and Smith proved that $(i)\zeta(R)$- semimodules $\zeta(M)$ and $\zeta(M^{'})$ are isomorphic if and only if $(ii)R$- lattices $RAD(M)$ and $RAD(M^{'})$ are isomorphic[9, theorem 3.2]. Also (iii) $Id(R)$-semimodules $RAD(M)$ and $RAD(M^{'})$ are isomorphic if and only if (iv) $Id(R)$- semimodules $\zeta(M)$ and $\zeta(M^{'})$ are isomorphic. In particular (i), (ii), (iii) and (iv) are equivalent. This idea causes the mentioned natural transformation to be a natural isomorphism between $\mathcal{R}$ and $\mathcal{Z}$(theorem 2.3). On the other hand it may causes the ambiguity that $\mathcal{R}$ or $\mathcal{Z}$ is full but it is not true. Moreover RAD and ZAR are isomorphic. McCasland, Moore and Smith have investigated, when some objects and morphisms are preserved or reflected by $\mathcal{R}$ and $\mathcal{Z}$. For example, the preserving epimorphisms and isomorphisms are given in lemma 8 of [7] and the conditions for preserving free objects is given in corollary 4.5 of [8]. Also conditions for reflecting the inclusions are given in lemmas 3.3 and 3.4. of [8] and conditions for reflecting epimorphism is given in lemma 8(ii) of [7]. According to these facts, the conditions for preserving and reflecting exact sequences by $\mathcal{R}$ and $\mathcal{Z}$ are given in corollary 2.7. In particular finite representations over domains are preserved by them(corollary 2.10).

2. Main results

We begin with a lemma which provides some elementary useful statements about radicals and varieties. The part (i) may be observed in corollary 1 to proposition 1 of [4] and the part (ii) is a consequence of proposition 2 of [5].

Lemma 2.1. *Let J be an ideal of R and $\{L_i \mid i \in I\}$ be a collection of submodules of M. Then*
(i)$rad(\Sigma_{i\in I}L_i) = rad(\Sigma_{i\in I}radL_i)$.
(ii)$rad(J(rad(L_i + L_j)) = rad(rad(JL_i) + rad(JL_j))$, for all i and j in I.
(iii)$V(L_i) = V(L_j)$ if and only if $rad(L_i) = rad(L_j)$, for all i, j in I.

Let Φ defined on Zariski spaces by $\Phi(\zeta(M)) = RAD(M)$ and on morphisms by $\Phi(f)(L) = radL'$, where $f : \zeta(M) \longrightarrow \zeta(M')$ defined by $f(V(L)) = V(L')$. Also Ψ defined on $Id(R)$-semimodules $RAD(M)$ by $\Psi(RAD(M)) = \zeta(M)$ and on morphisms by $\Psi(g)(V(L)) = V(L')$, where $g : RAD(M) \longrightarrow RAD(M')$ given by $g(rad(L)) = L'$. Clearly Φ and Ψ preserve identity and composition and are inverse together. Then

Theorem 2.2. *The full subcategories RAD and ZAR of the category $Id(R)$-semimodules are isomorphic.*

Let M be an R-module and $\tau_M : \zeta(M) \longrightarrow RAD(M)$ be the $Id(R)$-semimodule homomorphism that takes $\tau_M(V(L)) = rad(L)$. Then

Theorem 2.3. *$\tau : \mathcal{Z} \longrightarrow \mathcal{R}$ that assigns to each R-module M an $Id(R)$-semimodule homomorphism $\tau_M : \zeta(M) \longrightarrow RAD(M)$ is a natural isomorphism.*

In the next theorem the preserving of some special short exact sequences will be studied.

Theorem 2.4. *The followings hold;*
(i) $\mathcal{R}$(resp. $\mathcal{Z}$) preserves epimorphisms.
(ii)Let M' be finitely generated and $\mathcal{R}(f)$(resp. $\mathcal{Z}(f)$) be an epimorphism for some R-module homomorphism $f : M \longrightarrow M'$. Then f is an epimorphism.
(iii)Let M' be a semiprime module and $\mathcal{R}(f)$(resp. $\mathcal{Z}(f)$) be a monomorphism for some homomorphism $f : M \longrightarrow M'$. Then f is a monomorphism.
(iv)$\mathcal{R}$(resp. $\mathcal{Z}$)preserves isomorphisms.

Example 2.5. *(i) and (ii) show that, it is not necessary $\mathcal{R}(f)$ is injective, when f is injective even if $Ker(\mathcal{R}(f)) = rad0$. Moreover (iii) shows that the converse of (iii) in theorem 2.4 is not true.*
(i) The Z-module monomorphism $f : Z_2 \longrightarrow Z_4$ defined by $f(x) = 2x$ is considered. Since $RAD(Z_2) = \{0, Z_2\}$ and $RAD(Z_4) = \{\{\bar{0}, \bar{2}\}, Z_4\}$, we have $\mathcal{R}(f)(Z_2) = rad(f(Z_2)) = rad(0) = rad(f(0)) = \mathcal{R}(f)(0)$ while $Z_2 \neq 0$.
(ii)The Z- module homomorphism $f : Z \longrightarrow Z$ defined by $f(x) = 2x$ is given. Then $\mathcal{R}(f)(L) = rad(0)$ implies that $2L = f(L) \subseteq rad(f(L)) = rad(0)$. Therefore $Ker(\mathcal{R}(f)) = rad0$. But $\mathcal{R}(f)(Z) = rad(2Z) = rad(4Z) = \mathcal{R}(f)(2Z)$.
(iii)The non-injective Z-module epimorphism $f : Z_4 \longrightarrow Z_4/\{\bar{0}, \bar{2}\}$ is considered. Since $\mathcal{R}(f)(Z_4) = rad(Z_4/\{\bar{0}, \bar{2}\}) = Z_4/\{\bar{0}, \bar{2}\} \neq 0 = f(\{\bar{0}, \bar{2}\}) = f(rad(\{\bar{0}, \bar{2}\}) = \mathcal{R}(f)(\{\bar{0}, \bar{2}\})$, $\mathcal{R}(f)$ is injective. Also $\mathcal{R}(f)$ is surjective.

the following corollary is an immediate consequence of corollary 9 of [8] and theorem 2.3.

Corollary 2.6. *Let M be a semiprime R-module and $f : M \longrightarrow M'$ be an R-module epimorphism. Moreover let $Ker(\mathcal{R}(f)) = \{rad0\}$. Then both f and $\mathcal{R}(f)$ are injective.*

In theorem 2.4 was observed that functors $\mathcal{R}$ and $\mathcal{Z}$ preserve any exact sequences to forms $M' \longrightarrow M \longrightarrow 0$ and $0 \longrightarrow M' \longrightarrow M \longrightarrow 0$. the following corollary shows that it occurs in general, when the variables are semiprime. For example $\mathcal{R}$ and $\mathcal{Z}$ preserve(reflect) the exact sequences of free modules (of finite rank)over a domain R and (finite dimensional). In particular they preserve vector spaces.

Corollary 2.7. *Let $... \longrightarrow M_{i-1} \overset{f_{i-1}}{\longrightarrow} M_i \overset{f_i}{\longrightarrow} M_{i+1} \overset{f_{i+1}}{\longrightarrow} ...$ be an exact sequence of semiprime R-modules. Then $... \longrightarrow \mathcal{R}(M_{i-1}) \overset{\mathcal{R}(f_{i-1})}{\longrightarrow} \mathcal{R}(M_i) \overset{\mathcal{R}(f_i)}{\longrightarrow} \mathcal{R}(M_{i+1}) \overset{\mathcal{R}(f_{i+1})}{\longrightarrow} ...$ is an exact sequence of $Id(R)$-semimodules. Furthermore if each variable M_i is finitely generated by $\{m_{ij} \mid 1 \leq j \leq k_i\}$ in which every cyclic submodule $R(m_{ij})(1 \leq j \leq k_i)$ is a direct summand of M_i, then the converse is true. The similar statements hold when $\mathcal{R}$ is replaced by $\mathcal{Z}$.*

Proof. Using theorem 2.4, the verification of sufficient condition is easy. Conversely, $\mathcal{R}(f_i f_{i-1})(radRx) = \mathcal{R}(f_i)\mathcal{R}(f_{i-1})(radRx) = \mathcal{R}(0)(radRx) = \{rad0\}$ for all $x \in M_{i-1}$. It follows that $rad(f_i f_{i-1})(Rm_{ij}) = rad0 = 0$ for all generators m_{ij} of M_i. Thus $(f_i f_{i-1})(Rm_{ij}) = 0$. Consequently $Imf_{i-1} \subseteq Kerf_i$. Now let $f_i(Rm_{ij}) = 0$. Then $\mathcal{R}(f_i)(radRm_{ij}) = rad(f_i(Rm_{ij})) = rad0$. Thus there is a radical submodule L of M_{i-1} such that $rad(f_i)(L) = radRm_{ij}$. Using [3, lemma 6] $R(m_{ij})$ is a radical submodule. Now by [8, theorem 5.1]$f_i(L) = Rm_{ij}$ for all generatorsm_{ij} of M_i. It means $Kerf_{i-1} \subseteq Imf_i$.□
The following is a consequence of theorem 2.4 and corollary 2.6.

Corollary 2.8. *Let* $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *be a short exact sequence of modules in which* M' *and* M *are semiprime. Then* $\mathcal{R}(0) \longrightarrow \mathcal{R}(M') \xrightarrow{\mathcal{R}(f)} \mathcal{R}(M) \xrightarrow{\mathcal{R}(g)} \mathcal{R}(M'') \longrightarrow \mathcal{R}(0)$ *is a short exact sequences of* $Id(R)$*-semimodules. The converse is true when* M *and* M'' *are semiprime finitely generated and* M' *is semiprime. The similar statements hold when* $\mathcal{R}$ *is replaced by* $\mathcal{Z}$.

The subset $\Delta = \{V(N_1), ..., V(N_k)\}$ of $\zeta(M)$ is called (i)a subtractive generating set of M, if $\zeta(M) = \{V(L) \mid V(L) \supseteq V(\sum_{i=1}^{n} J_i N_i)$ for some ideals $J_i (1 \le i \le n)$ of $R\}$, (ii) a subtractive linearly independent set of M, if $V(0) \notin \Delta$ and also the inclusions $V(L) \supseteq V(N_i)$ and $V(L_k) \supseteq V(\sum_{i \neq k} J_k N_k)$ for some ideals J_k of R follow that $V(L) = V(0)$, (iii) a subtractive basis if it is a subtractive generating set and subtractive linearly independent set of M and there dose not exist a simple refinement of Δ. In [8, theorem 4.4] was shown that every finitely generated module has a subtractive basis. In particular every free module of finite rank n over a domain R has a subtractive basis of size n[8, corollary 4.5]. Furtheremore a free module F over a domain R with the basis $\{m_1, ..., m_n\}$ has a subtractive basis $\{V(Rm_1), ..., V(Rm_n)\}$[8, lemmas 3.5, 3.6 and theorem 4.3 and corollary 4.5]. Let R be a domain and F be a free R-module and $i : \Delta \longrightarrow \zeta(F)$ be the inclusion map. Also let an $Id(R)$-semimodule L and a map $f : \Delta \longrightarrow L$ are given. If $V(\sum_{k=1}^{n} I_k m_k) = V(\sum_{k=1}^{n} J_k m_k)$ for some ideals I_k and J_k, $(1 \le k \le n)$, then $V(\sum_{k=1}^{n} I_k m_k) \supset V(N_k) \cap V(\sum_{k=1}^{n} J_k m_k)$. Since Δ is subtractive linearly independent, $V(\sum_{k=1}^{n} I_k m_k) = V(0)$. It follows that $\sum_{k=1}^{n} I_k m_k = 0$. Then $I_k = 0$, for all $1 \le k \le n$. Hence $\bar{f}$ defined by $\bar{f}(V(\sum_{k=1}^{n} I_k m_k)) = (\sum_{k=1}^{n} I_k f(V(Rm_k)))$ is an $Id(R)$-semimodule homomorphism and clearly $\bar{f}i = f$. By theorem 2.3, it holds for radicals. Therefore $\mathcal{Z}(\mathcal{R})$ preserves free objects. Hence we include the following results.

Corollary 2.9. *Let* R *be a domain and* F *be a free* R*-module with a basis* $\delta = \{m_1, ..., m_n\}$. *Then* $\zeta(F)(RAD(F))$ *is a free object on* $\Delta = \{V(Rm_1), ..., V(Rm_n)\}$ $(\Delta = \{rad(Rm_1), ..., rad(Rm_n)\}$ *in the category* $SEMOD$.

Let M be a finitely generated module over a domain R. There is a free module F of finite rank and a finite free representation $0 \longrightarrow Ker\pi \xrightarrow{i} F \xrightarrow{\pi} M \longrightarrow 0$. Since $Ker\pi$ is semiprime, then by corollary 2.8 and corollary 2.9, $\zeta(0) \longrightarrow \zeta(Ker\pi) \xrightarrow{\zeta(i)} \zeta(F) \xrightarrow{\zeta(\pi)} \zeta(M) \longrightarrow \zeta(0)$ is a finite free representation of $Id(R)$-semimodules. Then

Corollary 2.10. *Let* M *be a finitely generated module over a domain* R. *Then* $\zeta(M)$ *is a homomorphic image of a free* $Id(R)$*-semimodule of finite rank. In* *particular, $\mathcal{Z}$ preserves finite free representation. The similar statement holds when $\zeta(M)$ and $\mathcal{Z}$ are replaced by $RAD(M)$ and $\mathcal{R}$ respectively.*

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