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PSEUDO VALUATION ON BCK-ALGEBRAS

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ABSTRACT. In this paper, we introduce the notion of pseudo valuations on a BCK-al - gebra and study its properties. We investigate the relation between pseudo valuations and ideals. We use pseudo valuation to define a pseudo metric on a BCK-algebra and prove that this pseudo metric induces a congruence relation on a BCK-algebra. We define the quotient algebra induced by this relation and prove that it is also a BCK-algebra.

1. INTRODUCTION

Imai and Iseki introduced the notion of a BCK-algebra in 1966(see [2], [3]). BCK-algebras have interesting algebraic properties (see [4], [5], [6]). Recently, D. Busneag defined valuations on residuated lattices ([1]).
Here, we review some definiens and theorems which are needed in the next section.

Definition 1.1. ([2], [3]) An algebra $(A, *, 0)$ of type $(2, 0)$ is called a *BCK-algebra*, if it satisfies the following conditions for any $x, y, z \in A$:
(BCK1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCK2) $x * 0 = x$,
(BCK3) $0 * x = 0$,
(BCK4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Theorem 1.2. [5,6] *Let $(A, *, 0)$ be a BCK-algebra. Define a binary operation $\leq$ on A by which $x \leq y$ if and only if $x * y = 0$ for any $x, y \in A$. Then $(A, \leq)$ is a partially ordered set with 0 as minimal element in the meaning that $x \leq 0$ implies $x = 0$ for any $x \in A$.*

Theorem 1.3. [6] *An algebra $(A, *, 0)$ of type $(2, 0)$ is a BCK-algebra if and only if there is a partially ordering $\leq$ on A such that the following conditions hold: for any $x, y, z \in A$:*
(1) $(x * y) * (x * z) \leq (z * y)$,
(2) $x * (x * y) \leq y$,
(3) $0 \leq x$,
(4) $x * y = 0$ if and only if $x \leq y$.

Theorem 1.4. [6] *Let $x, y, z \in A$ be any elements in BCK-algebra A. Then*
(1) $x \leq y$ implies $x * z \leq y * z$ and $z * x \leq z * y$,
(2) $(x * y) * z = (x * z) * y$,

(3) $x * y \leq z$ *if and only if* $x * z \leq y$,
(4) $x * (x * (x * y)) = x * y$,
(5) $0 * (x * y) = (0 * x) * (0 * y)$,
(6) $(x * y) * x = 0$,
(7) $x * (x * y) = 0$ *if and only if* $x = x * y$,
(8) $x, y \leq x * (x * y)$.

Definition 1.5 (*6*)**.** A subset I of A is called an ideal of A if
(1) $0 \in I$,
(2) $y \in I$, $x * y \in I$ imply $x \in I$ for any $x, y \in A$.

2. Pseudo valuation

Definition 2.1. A real function $\upsilon : A \rightarrow \Re$ is called a *pseudo-valuation on* A if
(v1) $\upsilon(0) = 0$,
(v2) $\upsilon(x) \leq \upsilon(x * y) + \upsilon(y)$ for all $x, y \in A$.
The pseudo-valuation υ is said to be a *valuation* if
(v3) $\upsilon(x) = 0$, implies $x = 0$.

If we interpret A as an implicational calculus, $y * x$ as the proposition $x \rightarrow y$ and 0 as truth (i.e., the proposition $x \rightarrow y$ is true if and only if $y * x = 0$) , the pseudo-valuation on A can be interpreted as falsity-valuation.

Theorem 2.2. *Let* υ *be a pseudo-valuation on* A*. Then*
(1) $0 \leq \upsilon(x)$,
(2) $x \leq y$ *implies* $\upsilon(x) \leq \upsilon(y)$,
(3) $\upsilon(x * z) \leq \upsilon(x * y) + \upsilon(y * z)$,
for all $x, y, z \in A$.

Example 2.3. Let $A = \{0, a, b, c\}$ be a BCK-algebra with table as below:

$*$	*0*	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define $\upsilon : A \rightarrow \Re$ by $\upsilon(0) = 0$, $\upsilon(a) = .5$, $\upsilon(b) = \upsilon(c) = 1$. Then υ is a pseudo valuation on A.

Lemma 2.4. *Let* I *be an ideal of* A *and* $t \in \Re_+$*. Define* $\upsilon_I : A \rightarrow \Re$ *by*

$$\upsilon_I(x) = \{ \begin{array}{cc} 0 & x \in I, \\ t & x \notin I. \end{array}$$

Then υ_I *is a pseudo-valuation on* A*.*

Lemma 2.5. *Let* υ *be a pseudo-valuation on* A*. Then* $I_\upsilon = \{x \in A : \upsilon(x) = 0\}$ *is an ideal of* A*.*

Proposition 2.6. *Let* I *be an ideal of* A*. Then* $I_{\upsilon_I} = I$*.*

Remark 2.7. The above proposition does not furnish a one to one correspondence between ideals and pseudo valuations, because two distinct pseudo valuations of a given BCK-algebra may induce the same ideal. Consider Example 2.3 and define $\upsilon_1(0) = \upsilon_1(c) = 0$, $\upsilon_1(a) = 2$, $\upsilon_2(b) = 3$ and $\upsilon_2(0) = \upsilon_2(c) = 0$, $\upsilon_2(a) = \upsilon_2(b) = 1$. Then υ_1, υ_2 are two pseudo valuations on A such that $I_{\upsilon_1} = \{0, c\} = I_{\upsilon_2}$.

Theorem 2.8. *Let $\upsilon : A \to \Re$ be a pseudo-valuation on A. Define $d_\upsilon : A \times A \to \Re$ by*

$$d_\upsilon(x, y) = \upsilon(x * y) + \upsilon(y * x)$$

for $(x, y) \in A \times A$. Then (A, d_υ) is a pseudo-metric on A.

Theorem 2.9. *d_υ is a metric on A if and only if υ is a valuation.*

Theorem 2.10. *Let υ be a valuation. In the metric space (A, d_υ), the function $* : A \to A$ is uniformly continuous.*

Definition 2.11. Let A be a BCK-algebra and υ be a pseudo valuation on A. Define the relation $\equiv_\upsilon$ by

$$x \equiv_\upsilon y \qquad \Longleftrightarrow \qquad d_\upsilon(x, y) = 0 \ \text{ for all } x, y \in A.$$

Theorem 2.12. *Let A be a BCK-algebra and υ be a pseudo valuation on A. Then $\equiv_\upsilon$ is a congruence relation on A.*

Definition 2.13. Let A be a BCK-algebra , υ be a pseudo valuation on A and$\equiv_\upsilon$ be a congruence relation which is defined in Definition 2.11. The set of all congruence classes $[x] = \{y \in A : x \equiv_\upsilon y\}$ is denoted by A/υ. On this set, we define $[x] * [y] = [x * y]$. The resulting algebra is denoted by A/υ and is called the *quotient algebra of* A with respect to the pseudo valuation υ.

Theorem 2.14. *The quotient algebra given in Definition 2.13, is a BCK-algebra and $d^*(\bar{x}, \bar{y}) = d(x, y)$ is a metric on A/υ. Moreover, quotient topology on A/υ coincide with the metric topology induced by d^*.*

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