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QUASI-PERMUTATION REPRESENTATIONS OF MAXIMAL PARABOLIC SUBGROUPS OF CHEVALLEY GROUPS

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Abstract. By a quasi-permutation matrix we mean a square matrix over the complex field C with non-negative integral trace. Thus every permutation matrix over C is a quasi-permutation matrix. For a given finite group G, the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational and the complex numbers are denoted by $q(G)$ and $c(G)$ respectively. Finally $r(G)$ denotes the minimal degree of a faithful rational valued complex character of G. The purpose of this paper is to calculate $c(G)$ and $r(G)$ for the maximal parabolic subgroups of Chevalley Groups.

1. Introduction

In [4] Wong defined a quasi-permutation group of degree n to be a finite group G of automorphisms of an n-dimensional complex vector space such that every element of G has non-negative integral trace. Also Wong studied the extent to which some facts about permutation groups to the quasi-permutation group situation. The author investigated further the analogy between permutation groups and quasi-permutation groups.

Let G be a finite linear group of degree n, that is, a finite group of automorphisms of an n-dimensional complex vector space. We shall say that G is a quasi-permutation group if the trace of every element of G is a non-negative rational integer. The reason for this terminology is that, if G is a permutation group of degree n, its elements, considered as acting on the elements of a basis of an n-dimensional complex vector space V, induce automorphisms of V forming a group isomorphic to G. The trace of the automorphism corresponding to an element x of G is equal to the number of letters left fixed by x, and so is a non-negative integer. Thus, a permutation group of degree n has a representation as a quasi-permutation group of degree n .

By a quasi-permutation matrix we mean a square matrix over the complex field C with non-negative integral trace. Thus every permutation matrix over C is a quasi-permutation matrix. For a given finite group G, let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation

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matrices. By a rational valued character we mean a complex character χ of G such that $\chi(g) \in Q$ for all $g \in G$. As the values of the characters of a complex representation are algebraic numbers, a rational valued character is in fact integer valued. A quasi-permutation representation of G is then simply a complex representation of G whose character values are rational and non-negative. The module of such a representation will be called a quasi-permutation module.We will call a homomorphism from G to $GL(n, Q)$ a rational representation of G and its corresponding character will be called a rational character of G. Let $r(G)$ denote the minimal degree of a faithful rational valued character of G. It is easy to see that for a finite group G the following inequalities hold

$$r(G) < c(G) \leq q(G).$$

It is easy to see that if G is a symmetric group of degree 6, then $r(G) = 5$ and $c(G) = q(G) = 6$, if G is a quaternion group of order 8, then $r(G) = 2, c(G) = 4$ and $q(G) = 8$. Our principal aim in this paper is to investigate these quantities and inequalities further.

2. Notation and preliminaries

Let $G = G_2(q)$ be the Chevalley group of type G_2 defined over K . We summarize some properties of the group. Let Σ be the set of roots of a simple Lie algebra of type G_2. In some fixed ordering the set of positive roots of Σ can be written as

$$\Sigma^+ = \{a, b, a+b, 2a+b, 3a+b, 3a+2b\},$$

and Σ consists of the elements of Σ^+ and their negatives. Moreover we denote the element $h(\chi)$ by $h(z_1, z_2, z_3)$, where $\chi(\xi_i) = z_i$ with $z_1 z_2 z_3 = 1$. Note that $a = \xi_2, b = \xi_1 - \xi_2$, and $\xi_1 + \xi_2 + \xi_3 = 0$. For simplicity of notation $h(x^i, x^j, x^{-i-j})$ is also denoted by $h_x(i, j, -i-j)$ for $x = \gamma, \theta, \omega$, etc. Let $X_r = \{x_r(t) \mid t \in K\}$ be the one-parameter subgroup corresponding to a root r. Set

$$H = \{h(z_1, z_2, z_3) \mid z_i \in K^\times, z_1 z_2 z_3 = 1\},$$

$$U = X_a X_b X_{a+b} X_{2a+b} X_{3a+b} X_{3a+2b},$$

$$B = HU \quad , \quad P = < B, \omega_a > \quad .$$

Then $B = N_G(U)$ is a Borel subgroup and P is the maximal parabolic subgroup containing B.

The character tables of parabolic subgroup

$$P = < B, \omega_a > = B \bigcup B\omega_a B$$

for different q are given in Tables $[A.4, A.6], [III, IV]$ of [1], [3] respectively.

Now we give algorithms for calculating of $r(G)$ and $c(G)$.

Definition 2.1. Let χ be a character of G such that, for all $g \in G$, $\chi(g) \in Q$ and $\chi(g) \geq 0$. Then we say that χ is a *non-negative rational valued character.*

In [2] we defined $d(\chi), m(\chi)$ and $c(\chi)$ [see Definition 3.4]. Here we can redefine it as follows:

Definition 2.2. Let χ be a complex charater of G such that $\ker\chi = 1$ and $\chi = \chi_1 + \cdots + \chi_n$ for some $\chi_i \in Irr(G)$. Then define

(i) $d(\chi) = \sum_{i=1}^{n} |\Gamma_i(\chi_i)|\chi_i(1)$,

(ii) $m(\chi) = \begin{cases} 0 & if\ \chi = 1_G, \\ |\min\{\sum_{i=1}^{n} \sum_{\alpha\in\Gamma_i(\chi_i)} \chi_i^\alpha(g) : g \in G\}| & otherwise, \end{cases}$

(iii) $c(\chi) = \sum_{i=1}^{n} \sum_{\alpha\in\Gamma_i(\chi_i)} \chi_i^\alpha + m(\chi)1_G$.

So

$$r(G) = \min\{d(\chi) : \ker\chi = 1\},$$

and

$$c(G) = \min\{c(\chi)(1) : \ker\chi = 1\}.$$

We can see all the following statements in [2] .

Corollary 2.3. *If* $\chi \in Irr(G)$, *then* $\sum_{\alpha\in\Gamma(\chi)} \chi^\alpha$ *is a rational valued character of* G . *Moreover* $c(\chi)$ *is a non-negative rational valued character of* G *and* $c(\chi) = d(\chi) + m(\chi)$.

Lemma 2.4. *Let* $\chi \in Irr(G), \chi \neq 1_G$. *Then* $c(\chi)(1) \geq d(\chi) + 1 \geq \chi(1) + 1$.

Lemma 2.5. *Let* $\chi \in Irr(G)$. *Then*
(i) $c(\chi)(1) \geq d(\chi) \geq \chi(1)$;
(ii) $c(\chi)(1) \leq 2d(\chi)$ *with equality if and only if* $Z(\chi)/ker\chi$ *is of even order* .

3. Quasi-permutation representations

Theorem 3.1. *Let* P *be a maximal parabolic subgroup of* $G_2(p^n); p \neq 3$. *Then*
(i) $r(P) = q^2(q-1)$,
(ii) $c(P) = q^3$,
(iii) $Lim_{q\longrightarrow\infty} \frac{c(P)}{r(P)} = 1$.

Table(I)

χ	$d(\chi)$	$c(\chi)(1)$
$\chi_7(k)$	$\geq q^2(q^2-1)$	$\geq q^3(q+1)$
$\theta_8(k)$	$\geq q^2(q-1)^2$	$\geq q^3(q-1)$
θ_7	$q^2(q-1)$	q^3
θ_8	$q^3(q-1)$	q^4

Theorem 3.2. *Let P be a maximal parabolic subgroup P of $G_2(3^n)$. Then*
(i) $r(P) = q(q-1)(q+2)$,
(ii) $c(P) = q^2(q+1)$,
(iii) $Lim_{q\longrightarrow\infty}\frac{c(P)}{r(P)} = 1$.

Table(II)

χ	$d(\chi)$	$c(\chi)(1)$
${}_P\chi_7(k)$	$\geq q^2(q^2-1)$	$\geq q^3(q+1)$
${}_P\theta_8(k)$	$\geq q^2(q-1)^2$	$\geq q^3(q-1)$
${}_P\theta_7$	$q^2(q-1)$	q^3
${}_P\theta_8$	$q^3(q-1)$	q^4
${}_P\theta_9$	$\frac{1}{2}q^2(q-1)^2$	$\frac{1}{2}q^3(q-1)$
${}_P\theta_{10}$	$\frac{1}{2}q^2(q-1)^2$	$\frac{1}{2}q^3(q-1)$
${}_P\theta_{11}$	$\frac{1}{2}q^2(q^2-1)$	$\frac{1}{2}q^3(q+1)$
${}_P\theta_{12}$	$\frac{1}{2}q^2(q^2-1)$	$\frac{1}{2}q^3(q+1)$

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