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f-MULTIPLICATION OF BINARY OPERATIONS AND DECOMPOSITIONAL GROUPS

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Abstract. The b-parts of real numbers were introduced and studied in [4], and b-addition of real numbers in [3]. The b-parts have many interesting number theoretic explanations, analytic and algebraic properties and the b-addition is b-decimal part of ordinary addition of two real numbers and is denoted $+_b$. It was shown that $(\mathbb{R}, +_b)$ is a semigroup (equivalently the b-decimal part function $(\ )_b$ satisfies the associative functional equation: $f(x+f(y+z)) = f(f(x+y)+z))$ and $\mathbb{R}_b = b[0,1)$ is its largest subgroup. As a generalization of this topic, decomposer and associative functions on binary systems (magmas), semigroups and groups are introduced and studied in [2]. If $(X, \cdot)$ is a binary system and f an arbitrary function from X to X, then another binary operation (in X) is defined by $x \cdot_f y = f(xy)$. The binary system $(X, \cdot_f)$ is a semigroup if and only if f is associative in $(X, \cdot)$. In [2] we solved associative equations in arbitrary groups and proved that the associative equation dose not have any nontrivial solutions in the simple groups. In this way all associative binary operations $\cdot_f$ for a group $(G, \cdot)$ are charateraized.
In this talk I discuss about the f-multiplication and the semigroup $(G, \cdot_f)$, where $f : G \to G$ is associative, and show that $(f(G), \cdot_f)$ is a group, namely f-decompositional group, and it is the largest subgroup of the semigroup $(G, \cdot_f)$. Also some other properties of f-decompositional groups will be considered.

1. Introduction

For any real number a denote by $[a]$ the largest integer not exceeding a and put $(a) = a - [a]$ (the decimal part of a). Now let b be a nonzero constant real number. For all real numbers a set

$$[a]_b = b[\frac{a}{b}] \ \ , \ \ (a)_b = b(\frac{a}{b}).$$

We call the notation $[a]_b$ *b-integer part of* a and $(a)_b$ *b-decimal part of* a. Also $[a]_b$ and $(a)_b$ are called *b-parts of* a.
Now let a , b are positive integers. By the division algorithm we have $a = bq+r$

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where q , r are integers and $0 \leq r < b$, so $(a)_b = (bq+r)_b = (r)_b = r$. It means that $(a)_b$ is the same remainder of the division of a by b. This is an important fact that leads us to several properties of b-parts.
The functions $(\ )_b$ and $[\ ]_b$ are idempotent, their compositions are zero, and $(\ )_b$ satisfies the following functional equations

$$f(f(x)+y-f(y)) = f(x)\ ,\ f(x+y-f(y)) = f(x)\ ,\ f(x+f(y+z)) = f(f(x+y)+z).$$

Recall that $\overline{\mathbb{Z}}_n = \{[0],[1],[2],...,[n-1]\} = \mathbb{Z}/n\mathbb{Z}$ is the group of integers modulo n and $\mathbb{Z}_n = \{0,1,2,3,4,...,n-1\}$ is the group of the least nonnegative residues modulo n. As we know $\overline{\mathbb{Z}}_n \cong \mathbb{Z}_n$. It is interest to know that the addition of $x, y \in \mathbb{Z}_n$ is the same $(x+y)_n = x +_n y$. This fact together with the generalized division algorithm and b-decimal part function lead us to a new binary operation on $\mathbb{R}$ and b-bounded groups. Fix a real number $b \neq 0$. Put $(R)_b = \{(r)_b | r \in R\}$ and $[R]_b = \{[r]_b | r \in R\}$ for every subset R of real numbers. For every $x, y \in \mathbb{R}$, we set

$$x +_b y = (x+y)_b$$

and call it x *b-addition* y. We have

$$(x +_b y) +_b z = (x+y+z)_b = x +_b (y +_b z),$$

for all $x, y, z \in \mathbb{R}$. Therefore $(\mathbb{R}, +_b)$ is a semigroup but it is not a monoid. Because if $x +_b e = x$, then $x = (x+e)_b \in b[0,1)$. In fact a sub-semigroup S of $(\mathbb{R}, +_b)$ is a monoid if and only if $0 \in S \subseteq \mathbb{R}_b := (\mathbb{R})_b = b[0,1)$. Therefore a necessary condition for a subset of real numbers to be a group with the b-addition is to be b-bounded set (subset of $\mathbb{R}_b$). Since $x +_b (b-x) = (b-x) +_b x = 0$, then $(\mathbb{R}_b, +_b)$ is the largest sub-semigroup of the semigroup $(\mathbb{R}, +_b)$ that is a group. So we call it *the reference b-bounded group.*

Definition 1.1. We call each subgroup of $(\mathbb{R}_b, +_b)$ *b-bounded group.* Every 1-bounded group is called a *decimal group.*

Example 1.2. If $n \in \mathbb{Z} \setminus \{0\}$, then $(\mathbb{Z})_n$ is a n-bounded group and it is the same $\mathbb{Z}_n$, if $n > 0$. For every positive integer $n \geq 2$ put $\mathbb{D}_n = \{0, \frac{1}{n}, \cdots, \frac{n-1}{n}\}$, then $\mathbb{D}_n$ is a decimal group (we call it n-decimal group). It has many interesting number theoretic and algebraic properties.

In this talk we consider X as a set with the binary operation "$\cdot$" (the binary system $(X, \cdot)$ or magma) where the product of $x, y \in X$ is denoted by xy, $(S, .)$ as a semigroup and also $(G, .)$ as a group. If the multiplication "$\cdot$" is commutative ($\cdot(x,y) = \cdot(y,x)$ for every $x, y \in X$), then it is denoted by $+$ and is called addition. The two sided unit element of X and S (if exists) is denoted by 1 and for G by e.
If $f : G \longrightarrow G$, then we define the function f^- by $f^-(x) = [f(x)]^{-1}$, for every $x \in G$, and put $f^* = \iota_G \cdot f^-$, $f_* = f^- \cdot \iota_G$ (ι_G is the identity function on G). A function f from X to X is called associative if $f(xf(yz)) = f(f(xy)z)$, for eery $x, y, z \in X$.

1.1. **Direct product of subsets.** As we know if Δ, Ω are subsets of X, then $\Delta\Omega = \cdot(\Delta \times \Omega)$. Now we call the product direct and denote by $\Delta \cdot \Omega$ ($\Delta \dot{+} \Omega$, for additive notation) if $\cdot|_{\Delta\times\Omega}$ is injective. Note that $Y = \Delta \cdot \Omega$ means $Y = \Delta\Omega$ and the product $\Delta\Omega$ is direct. We call $Y = \Delta \cdot \Omega$ a factorization of Y by Δ

and Ω and call Δ $[\Omega]$ a left [right] factor of Y. If $\Delta\Omega = \Delta \cdot \Omega$ $[\Delta + \Omega = \Delta \dot{+} \Omega]$ and $\Delta \cap \Omega = \{1\}$, then we use the notation $\Delta : \Omega$ $[\Delta \ddot{+} \Omega]$ and call it *standard direct product [addition]* of Δ and Ω.
If Δ and Ω are finite subsets of X, then

$$\Delta\Omega = \Delta \cdot \Omega \Leftrightarrow |\Delta\Omega| = |\Delta||\Omega|.$$

If $\Delta\Omega = \Delta \cdot \Omega$ and $\Delta \cap \Omega$ has an element such that commutes with every elements of $\Delta \cap \Omega$, then $|\Delta \cap \Omega| = 1$. Especially if $\Delta\Omega = \Delta \cdot \Omega$ and $1 \in \Delta \cap \Omega$, then $\Delta\Omega = \Delta : \Omega$.
It is easy to see that if S is monoid and $\emptyset \neq \Delta, \Omega \subseteq S$ and Δ^{-1}, Ω^{-1} exit, then

$$\Delta\Omega = \Delta \cdot \Omega \Leftrightarrow \Delta^{-1}\Delta \cap \Omega\Omega^{-1} = \{1\}.$$

Example 1.3. The set $\mathbb{N}^* = \mathbb{Z}^+$ $[\mathbb{N} = \mathbb{N}^* \cup \{0\}]$ is not a factor of $\mathbb{Z}$, because if $\mathbb{Z} = \mathbb{N}^* \dot{+} \Omega$, then $\Omega - \Omega = (\mathbb{N}^* - \mathbb{N}^*) \cap (\Omega - \Omega) = \{0\}$ that is impossible. But $\mathbb{Z} = \mathbb{Z}_e \ddot{+} \{0, 1\} = \mathbb{Z}_o \ddot{+} \{0, 1\}$. Also $S_3 =< \sigma >:< \tau >$ (where $\sigma^3 = \tau^2 = 1$) but we do not have any (nontrivial) representation $S_3 = A \times B$, where A and B are normal subgroups of S_3. Also we have $\mathbb{R} = \mathbb{R}_b \ddot{+} < b >$.

Consider the binary system $(X, \cdot)$ and a function f from X to X. We get another binary operation in X by defining $x \cdot_f y = f(xy)$. In fact $\cdot_f = fo\cdot$, hence we call it *f-multiplication of "$\cdot$"*. Clearly $(X, \cdot_f)$ is a semigroup if and only if f is associative (e.g. if f is constant function). Also it is clear that $Z(X, \cdot) \subseteq Z(X, \cdot_f)$. But in general $Z(X, \cdot_f) \nsubseteq Z(X, \cdot)$, for if $X = G$ is a group with the center $\{e\}$ and f is a non-standard associative function on X (i.e. $f(e) \neq e$), then $\{e, f(e)\} \subseteq Z(X, \cdot_f)$. Now we want to generalize the conception b-addition and also b-bounded groups from real numbers system to arbitrary groups and state their fundamental theorem.

2. Main results

Theorem 2.1. *If $f : G \to G$ is associative, then $(f(G), \cdot_f)$ is a group with the identity element $f(f(e)^{-2})$ and it is the largest subgroup of the semigroup $(G, \cdot_f)$. Furthermore if $f(e) = e$ (the standard case), then $f^*(G) = f_*(G) \trianglelefteq G$ and*

$$G = f^*(G) : f(G) = f(G) : f^*(G) \ , \ \frac{G}{f^*(G)} \approxeq (f(G), \cdot_f).$$

Therfore we call $(f(G), \cdot_f)$ *f-decompositional group of G.*

Theorem 2.2. *(Unique representation of subgroups by standard decomposer functions). Let $f : G \to G$ be a function and $\Omega_f = f(G)$ and $\Delta_f = f^*(G)$.*
(a) *If $G = \Delta_f : \Omega_f$, then every subgroup H containing Δ_f $[\Omega_f]$ has the unique representation $H = \Delta_f : \Omega'$ $[H = \Delta' : \Omega_f]$, where $\Omega' \subseteq \Omega_f$ and $\Delta' \subseteq \Delta_f$.*
(b) *If $G =< g >: \Omega_g$ ($\Omega_g = \Omega_f$) then every subgroup H containing (the fix element) g has the unique representation $H =< g >: \Omega'$, where $\Omega' \subseteq \Omega_f$.*

Theorem 2.3. *If f is standard associative, then*
i) *Every subgroup H containing Δ_f has the unique representation $H = \Delta_f : \Omega'$, where Ω' is decompositional group.*
ii) *All subgroups of G containing Δ_f and all f-decompositional groups can be*

put in $1-1$ *correspondence.*
iii) *If* $\Delta_f \subseteq H \leq G$, *then* ($\Delta_f \trianglelefteq H$ *and*) $\frac{H}{\Delta_f} = \frac{H}{f^*(H)} \approxeq f(H) = \Omega_H = \Omega_f \cap H$.

Example 2.4. (i) If R is an additive sub-group of $\mathbb{R}$ and $b \in R \setminus \{0\}$, then $(R)_b = R \cap \mathbb{R}_b$, and $\frac{R}{<b>}$ is isomorphic to a decimal group D^b and

$$\frac{R}{<b>} = \frac{R}{[R]_b} \cong (R)_b = R \cap \mathbb{R}_b \cong \frac{1}{b}(R)_b = D^b.$$

Specially if $1 \in R$, then $\frac{R}{\mathbb{Z}} \cong (R)_1 = R \cap [0,1)$, so $\frac{\mathbb{R}}{\mathbb{Z}} \cong ([0,1), +_1)$.
(ii) If D is a decimal group, then

$$\frac{b\mathbb{Z} + bD}{b\mathbb{Z}} \cong D.$$

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