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OBSTINATE PSEUDO-BCK IDEAL OF PSEUD-BCK ALGEBRAS

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Abstract. The notion of obstinate pseudo-BCK ideals is introduced and some related properties are investigated. We investigate the relations between this pseudo ideals and several pseudo-BCK ideals. We give the following characterization theorem of obstinate ideals in pseudo-BCK algebras: Let I be an pseudo ideal of pseudo-BCK algebra X. Then I is obstinate pseudo-BCK ideal if and only if I is pseudo maximal, pseudo irreducible, prime, positive implicative or implicative. In this talk we discuss about a relation between obstinate pseudo-BCK ideals and other above pseudo-BCK ideals.

1. Introduction

BCK-algebras are important logical algebras introduced by Iseki in 1966(see[3]). Since then this class of algebras has been intensively studied by many researchers. In particular, emphasis have been put on the ideal theory of BCK-algebras. In 1987, S.K.Goel and A.K.Arora[2] first introduced the concept of obstinate ideals in BCK-algebras. Georgescu and Iorgulescu [1] introduced the notion of pseudo-BCK algebras and studied their properties. In [4], Y. B. Jun, gave characterizations of pseudo-BCK ideals such as positive implicative pseudo-BCK ideals. In this talk we introduced the notion of obstinate pseudo-BCK ideals and show the relation between these ideals and (implicative, positive implicative, commutative, maximal, prime and irreducible) pseudo BCK-ideals of pseudo-BCK algebras. Recall that a pseudo-BCK algebra is a structure $(X, \leq, *, \diamond, 0)$, where $\leq$, $*$ and $\diamond$ are binary operations on X and 0 is an element of X verifying the following axioms for all $x, y, z \in X$.

1): $(x * y) \diamond (x * z) \leq z * y$, $(x \diamond y) * (x \diamond z) \leq z \diamond y$,
2): $x * (x \diamond y) \leq y$, $x \diamond (x * y) \leq y$,

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3): $x \leq x$,
4): $0 \leq x$,
5): $x \leq y$ and $y \leq x$ imply $x = y$,
6): $x \leq y \Leftrightarrow x * y = 0 \Leftrightarrow x \diamond y = 0$.

Note that every pseudo-BCK algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$ is a BCK-algebra. A pseudo-BCK algebra X is said to be $\wedge -$ *commutative* (resp. $\cap -$ *commutative*) if it satisfies for all x, y in X, $x \diamond (x * y) = y \diamond (y * x)$ i.e. $y \wedge x = x \wedge y$ (resp. $x * (x \diamond y) = y * (y \diamond x)$ i.e. $y \cap x = x \cap y$). A nonempty subset I of pseudo-BCK algebra X is said to be an pseudo-BCK ideal of X if it satisfies: (i) $0 \in I$, and (ii) $x * y$, $x \diamond y \in I$ and $y \in I$ imply $x \in I$.

2. Concepts on pseudo-BCK ideals

Definition 2.1. A pseudo-BCK algebra X is called to be *Positive implicative* if for all x, y and z in X, $(x * z) \diamond (y * z) = (x * y) \diamond z$, or for all x, y and z in X, $(x \diamond z) * (y \diamond z) = (x \diamond y) * z$.

Example 2.2. The $X = \{0, a, b, c\}$ equipped with the operations $*$, $\diamond$ given by the following tables is a positive implicative pseudo-BCK algebra.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	c	0

$\diamond$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	a	0

Definition 2.3. Suppose $(X, *, \diamond, 0)$ is a pseudo-BCK algebra. A nonempty subset I of X is said to be an *positive implicative pseudo-BCK ideal*, if it satisfies the following conditions for all $x, y, z \in X$,
(i) $0 \in I$,
(ii) $(x * y) \diamond z \;,\; y \diamond z \in I \implies x \diamond z \in I$,
(iii) $(x \diamond y) * z \;,\; y * z \in I \implies x * z \in I$.

Example 2.4. In example 2.2 the pseudo-BCK ideals of X are $\{0\}$, $\{0, a\}$, $\{0, b\}$ and X. $I = \{0, a\}$ is a positive implicative pseudo-BCK ideal. Also the other ideals of X are positive implicative pseudo-BCK ideal.

Definition 2.5. A pseudo-BCK algebra X is called *implicative pseudo-BCK algebra* if for all $x, y \in X$, $x = x \diamond (y * x) = x * (y \diamond x)$.

Definition 2.6. Suppose $(X, *, \diamond, 0)$ is a pseudo-BCK algebra. A nonempty subset I of X is said to be an *implicative pseudo-BCK ideal* if it satisfies the following conditions, for all $x, y, z \in X$,
(i) $0 \in I$,
(ii) $(x * (y \diamond x)) * z \;,\; z \in I \implies x \in I$,
(iii) $(x \diamond (y * x)) \diamond z \;,\; z \in I \implies x \in I$.

A pseudo-BCK algebra X is called *with condition* (K), if $(x * z) \diamond (y * z) \leq x * y$ *and* $(x \diamond z) * (y \diamond z) \leq x \diamond y$ for all $x, y \in X$.

Theorem 2.7. *An implicative pseudo-BCK ideal of pseudo-BCK algebra with condition (K) is a positive implicative pseudo-BCK ideal, but the invers is not true.*

Definition 2.8. A pseudo-BCK algebra X is called to be *commutative pseudo-BCK algebra* if it satisfies for all x, y and z in X,
(a) $x \diamond (x * y) = y \diamond (y * x)$ and (b) $x * (x \diamond y) = y * (y \diamond x)$.

Definition 2.9. A nonempty subset I of X is called *commutative pseudo-BCK ideal* if it satisfies, for all x, y, z in X, *the following conditions*

- **(i):** $0 \in I$,
- **(ii):** $(x * y) \diamond z \in I \quad and \quad z \in I \quad \Longrightarrow \quad x * (y \diamond (y * x)) \in I$,
- **(iii):** $(x \diamond y) * z \in I \quad and \quad z \in I \quad \Longrightarrow \quad x \diamond (y * (y \diamond x)) \in I$,

Theorem 2.10. *Let X be a pseudo BCK-algebra with condition (K), then a nonempty subset I of X is an implicative pseudo-BCK ideal if and only if, it is both a commutative pseudo-BCK ideal and positive implicative pseudo-BCK ideal.*

3. Obstinate pseudo-BCK ideal

Definition 3.1. A proper pseudo-BCK ideal I of X is called *obstinate* pseudo-BCK ideal if for any $x, y \in X$,

- **(a):** $x, y \notin I \Rightarrow x * y \in I$ and $y * x \in I$,
- **(b):** $x, y \notin I \Rightarrow x \diamond y \in I$ and $y \diamond x \in I$.

Theorem 3.2. *Let I be a pseudo-BCK ideal of X. Then the following are equivalent:*

- **(a):** *I is obstinate pseudo-BCK ideal,*
- **(b):** *I is positive pseudo-BCK ideal and maximal pseudo-BCK ideal,*
- **(c):** *I is implicative pseudo-BCK ideal and maximal pseudo-BCK ideal.*

Definition 3.3. A pseudo-BCK ideal I of a pseudo-BCK algebra X is called *maximal pseudo-BCK ideal* if it is proper and no pseudo-BCK ideal of X strictly contains I, i.e., for each pseudo-BCK ideal $J \neq I$, if $I \subseteq J$ then $J = X$.

Definition 3.4. *A proper pseudo-BCK ideal I of a pseudo-BCK algebra X is called irreducible pseudo-BCK ideal, if $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$ for any pseudo ideals I_1 and I_2 of X.*

Definition 3.5. Let I be a proper pseudo-BCK ideal of a pseudo-BCK algebra X. Then I is called *prime pseudo-BCK ideal*, if $x \wedge y \in I$ implies $x \in I$ or $y \in I$ for any $x, y \in X$. $(x \wedge y := y \diamond (y * x))$

Theorem 3.6. *For a pseudo ideal I of a pseudo-BCK algebra X, the following are equivalent.*

- **(a):** *I is obstinate pseudo-BCK ideal.*
- **(b):** *I is maximal pseudo-BCK ideal.*
- **(c):** *I is prime pseudo-BCK ideal.*
- **(d):** *I is irreducible pseudo-BCK ideal.*

Definition 3.7. Let X and Y be pseudo-BCK algebras. A mapping $f : X \to Y$ is called a *pseudo homomorphism* if $f(x * y) = f(x) * f(y)$ and $f(x \diamond y) = f(x) \diamond f(y)$ for all x, y in X.

Theorem 3.8. *Let I be a proper pseudo ideal of a pseudo-BCK algebra X. Then for any pseudo-BCK algebra X' there exists $f \in Hom(X, X')$ such that $ker(f) = I$, if and only if, I is obstinate pseudo-BCK ideal.*

Theorem 3.9. *Let X, Y and Z are pseudo-BCK algebras. Let $h : X \to Y$ be an epimorphism and $g \in Hom(X, Z)$. If $ker(h) \subset ker(g)$ then there exists a unique homomorphism $f : Y \to Z$ such that $foh = g$.*

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