

EDGE-TRANSITIVE ELEMENTARY ABELIAN REGULAR COVER OF Q_3

MOHSEN LASHANI

Faculty of Mathematics
Iran University of Science and Technology
Narmak, Tehran 16844, Iran lashani@iust.aci.ir
(Joint work with Mehdi Alaeiyan and M.K. Hosseinipoor)

ABSTRACT. A simple undirected graph is said to be semisymmetric if it is regular and edge-transitive but not vertex-transitive. Let $p \geq 11$ be a prime. In this paper, it is proved that, every cubic edge-transitive elementary abelian regular cover of Q_3 is vertex-transitive.

1. INTRODUCTION

In this paper we consider an undirected finite connected graph without loops or multiple edges. For a graph Γ , we denote by $V(\Gamma)$, $E(\Gamma)$ and $\text{Aut}(\Gamma)$ its vertex set, edge set and automorphism group, respectively. For $u, v \in V(\Gamma)$, denote by uv the edge incident to u and v in Γ , and by $N_\Gamma(u)$ the *neighbourhood* of u in Γ , that is, the set of vertices adjacent to u in Γ . A graph $\tilde{\Gamma}$ is called a *covering* of a graph Γ with projection $p : \tilde{\Gamma} \rightarrow \Gamma$ if there is a surjection $p : V(\tilde{\Gamma}) \rightarrow V(\Gamma)$ such that $p|_{N_{\tilde{\Gamma}}(\tilde{v})} : N_{\tilde{\Gamma}}(\tilde{v}) \rightarrow N_\Gamma(v)$ is a bijection for any vertex $v \in V(\Gamma)$ and $\tilde{v} \in p^{-1}(v)$. Let N be a subgroup of $\text{Aut}(\Gamma)$ such that N is intransitive on $V(\Gamma)$. The quotient graph Γ/N induced by N is defined as the graph such that the set Σ of N -orbits in $V(\Gamma)$ is the vertex set of Γ/N and $B, C \in \Sigma$ are adjacent if and only if there exist $u \in B$ and $v \in C$ such that $\{u, v\} \in E(\Gamma)$. A covering $\tilde{\Gamma}$ of Γ with a projection p is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism group $\text{Aut}(\tilde{\Gamma})$ such that graph Γ is isomorphic to the quotient graph $\tilde{\Gamma}/K$, say by h , and the quotient map $\tilde{\Gamma} \rightarrow \tilde{\Gamma}/K$ is the composition ph of p and h (for the purpose of this paper, all functions are composed from left to right). If K is cyclic or elementary abelian then $\tilde{\Gamma}$ is called a *cyclic* or an *elementary abelian covering* of Γ , and if $\tilde{\Gamma}$ is connected K becomes the covering transformation group.

If a subgroup G of $\text{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$, we say that Γ is *G-vertex-transitive*, *G-edge-transitive* and *G-arc-transitive*, respectively. In the special case, when $G = \text{Aut}(\Gamma)$ we say that Γ is vertex-transitive,

2000 Mathematics Subject Classification: 05C25, 20B25.

keywords and phrases: semisymmetric graphs, symmetric graphs, regular coverings.

edge-transitive and arc-transitive (or *symmetric*), respectively. A regular G -edge-transitive but not G -vertex-transitive graph will be referred to as a G -*semisymmetric graph*. In particular, if $G = \text{Aut}(\Gamma)$, then the graph Γ is said to be semisymmetric.

The study of semisymmetric graphs was initiated by Folkman [2]. It is given a classification of semisymmetric graphs of order $2pq$ in [1], where p and q are distinct primes.

Let Γ be a graph and K be a finite group. By a^{-1} we mean the reverse arc to an arc a . A *voltage assignment* (or, K -*voltage assignment*) of Γ is a function $\phi : A(\Gamma) \rightarrow K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(\Gamma)$. The values of ϕ are called *voltages*, and K is the *voltage group*. The graph $\Gamma \times_\phi K$ derived from a voltage assignment $\phi : A(\Gamma) \rightarrow K$ has vertex set $V(\Gamma) \times K$ and edge set $E(\Gamma) \times K$, so that an edge (e, g) of $\Gamma \times K$ joins a vertex (u, g) to $(v, \phi(a)g)$ for $a = (u, v) \in A(\Gamma)$ and $g \in K$, where $e = uv$.

Clearly, the derived graph $\Gamma \times_\phi K$ is a covering of Γ with the first coordinate projection $p : \Gamma \times_\phi K \rightarrow \Gamma$, which is called the *natural projection*. By defining $(u, g')^g = (u, g'g)$ for any $g \in K$ and $(u, g') \in V(\Gamma \times_\phi K)$, K becomes a subgroup of $\text{Aut}(\Gamma \times_\phi K)$ which acts semiregularly on $V(\Gamma \times_\phi K)$. Therefore, $\Gamma \times_\phi K$ can be viewed as a K -*covering*. For each $u \in V(\Gamma)$ and $uv \in E(\Gamma)$, the vertex set $\{(u, g) \mid g \in K\}$ is the fibre of u and the edge set $\{(u, g)(v, \phi(a)g) \mid g \in K\}$ is the fibre of uv , where $a = (u, v)$. Conversely, each regular covering $\tilde{\Gamma}$ of Γ with a covering transformation group K can be derived from a K -voltage assignment. Given a spanning tree T of the graph Γ , a voltage assignment ϕ is said to be T -*reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [5] showed that every regular covering $\tilde{\Gamma}$ of a graph Γ can be derived from a T -reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of Γ . It is clear that if ϕ is reduced, the derived graph $\Gamma \times_\phi K$ is connected if and only if the voltages on the cotree arcs generate the voltages group K .

Let $\tilde{\Gamma}$ be a K -covering of Γ with a projection p . If $\alpha \in \text{Aut}(\Gamma)$ and $\tilde{\alpha} \in \text{Aut}(\tilde{\Gamma})$ satisfy $\tilde{\alpha}p = p\alpha$, we call $\tilde{\alpha}$ a *lift* of α , and α the *projection* of $\tilde{\alpha}$. Concepts such as a lift of a subgroup of $\text{Aut}(\Gamma)$ and the projection of a subgroup of $\tilde{\Gamma}$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in $\text{Aut}(\tilde{\Gamma})$ and $\text{Aut}(\Gamma)$ respectively. In particular, if the covering graph $\tilde{\Gamma}$ is connected, then the covering transformation group K is the lift of the trivial group, that is $K = \{\tilde{\alpha} \in \text{Aut}(\tilde{\Gamma}) : p = \tilde{\alpha}p\}$. Clearly, if $\tilde{\alpha}$ is a lift of α , then $K\tilde{\alpha}$ are all the lifts of α .

2. MAIN RESULTS

Lemma 2.1. *Suppose that Γ is a connected semisymmetric cubic graph of order $8p^n$. Then Γ is a connected N -regular covering of Q_3 such that the subgroup of $\text{Aut}(Q_3)$ generated by α and β lifts, where $|N| = p^n$*

Lemma 2.2. *Let $N \cong \mathbb{Z}_p^n$ and suppose that $\Gamma = Q_3 \times_\phi \mathbb{Z}_p^n$ is a connected \mathbb{Z}_p^n -regular covering of Q_3 . If the subgroup of $\text{Aut}(Q_3)$ generated by α and β can be lifted then Γ is symmetric.*

Theorem 2.3. *Let $p \geq 11$ be a prime. Then every cubic edge-transitive elementary abelian regular cover of Q_3 is vertex-transitive.*

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