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SOME PROPERTIES ABOUT NON-ABELIAN TENSOR PRODUCT OF LIE ALGEBRAS

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Abstract. This paper is devoted to study the structure of the non-abelian tensor product of Lie algebras and the connection of nilpotency and solvability between the Lie algebras and their tensor products. In addition, some certain bounds for the order and the tensor products and the Schur multiplier of nilpotent Lie algebras will be presented.

1. Introduction

All Lie algebras are considered over a fixed field F and $[\ ,\ ]$ denotes the Lie bracket.

Let L, K be two Lie algebras. By an action of L on K we mean an F-bilinear map $L \times K \to K, (l, k) \mapsto {}^{l}k$ satisfying

$$^{[l,l']}k = {}^{l}({}^{l'}k) - {}^{l'}({}^{l}k) \text{ and } {}^{l}[k, k'] = [{}^{l}k, k'] + [k, {}^{l}k'],$$

for all $l, l' \in L$ and $k, k' \in K$. Clearly, if L is a subalgebra of some Lie algebra P and K is an ideal in P, then the Lie multiplication in P induces an action of L on K. In fact, $l \in L$ acts on $k \in K$ by ${}^{l}k = [l, k]$.

Let L and K be Lie algebras acting on each other, and on themselves by Lie multiplications. Then

(i) These actions are said to be *compatible* if ${}^{({}^{k}l)}k' = {}^{k'}({}^{l}k)$ and ${}^{({}^{l}k)}l' = {}^{l'}({}^{k}l)$, for all $l, l' \in L$ and $k, k' \in K$. It is obvious that if L and K are both ideals of some Lie algebra, then the Lie multiplication gives rise to compatible actions.

(ii) For each Lie algebra Q we call a bilinear function $\varphi : K \times L \to Q$ a *Lie pairing* if for all $l, l' \in L$ and $k, k' \in K$,

$$\begin{aligned}
\varphi([l, l'], k) &= \varphi(l, {}^{l'}k) - \varphi(l', {}^{l}k),\\
\varphi(l, [k, k']) &= \varphi({}^{k'}l, k) - \varphi({}^{k}l, k'),\\
\varphi({}^{k}l, {}^{l'}k') &= -[\varphi(l, k), \varphi(l', k')].
\end{aligned}$$

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(iii) The *non-abelian tensor product* $L \otimes K$ is the Lie algebra generated by the symbols $l \otimes k$ $(l \in L, k \in K)$ with the following defining relations

$$\begin{aligned}
&\text{(a) } c(l \otimes k) = cl \otimes k = l \otimes ck,\\
&\text{(b) } (l + l') \otimes k = l \otimes k + l' \otimes k,\\
&\quad\ l \otimes (k + k') = l \otimes k + l \otimes k',\\
&\text{(c) } {}^{l}l' \otimes k = l \otimes {}^{l'}k - l' \otimes {}^{l}k,\\
&\quad\ l \otimes^{k} k' = {}^{k'}l \otimes k - {}^{k}l \otimes k',\\
&\text{(d) } [(l \otimes k), (l' \otimes k')] = -{}^{k}l \otimes {}^{l'}k'
\end{aligned}$$

for all $c \in F, l, l' \in L$ and $k, k' \in K$. If $L = K$ and all actions are given by Lie multiplication, then $L \otimes L$ is called the *non-abelian tensor square*. One notes that the non-abelian tensor square of a given Lie algebra always exists.

Lie pairings allow us to determine homomorphic images of $L \otimes K$ as follows:

Lemma 1.1. *For every Lie algebra Q and each Lie pairing $\varphi : L \times K \to Q$, there exists a unique homomorphism $\varphi^* : L \otimes K \to Q$ such that $\varphi^*(l \otimes k) = \varphi(l, k)$ for all $l \in L, k \in K$.*

Proposition 1.2. *Let L and K be Lie algebras which act compatibly on each other. Then*

(i) *The Lie algebras L and K acting on $L \otimes K$ so that*

$${}^{l'}(l \otimes k) = {}^{l'}l \otimes k + l \otimes {}^{l'}k \quad , \quad {}^{k'}(l \otimes k) = {}^{k'}l \otimes k + l \otimes {}^{k'}k$$

for all $l, l' \in L$ and $k, k' \in K$.

(ii) *Suppose $\sigma_1 : L \to P$ and $\sigma_2 : K \to Q$ are Lie homomorphisms, where P and Q act on each other in a compatible way, and σ_1, σ_2 preserve the actions in the sense that*

$$\sigma_1({}^{k}l) = {}^{\sigma_2(k)}(\sigma_1(l)) \text{ and } \sigma_2({}^{l}k) = {}^{\sigma_1(l)}(\sigma_2(k)),$$

for all $l \in L$ and $k \in K$. Then there is a unique homomorphism

$$\sigma_1 \otimes \sigma_2 : L \otimes K \to P \otimes Q$$

such that $\sigma_1 \otimes \sigma_2(l \otimes k) = \sigma_1(l) \otimes \sigma_2(k)$ for all $l \in L, k \in K$. Furthermore, if σ_1, σ_2 are onto then so is $\sigma_1 \otimes \sigma_2$.

(iii) *If L and K act trivially on each other then $L \otimes K \cong L^{\text{ab}} \otimes K^{\text{ab}}$, where $L^{\text{ab}} = L/L^2$ and $K^{\text{ab}} = K/K^2$*

(iv) *There is an isomorphism $L \otimes K \xrightarrow{\cong} K \otimes L$, $l \otimes k \mapsto -k \otimes l$.*

(v) *There is a F-module surjection $L \underset{\text{mod}}{\otimes} K \to L \otimes K$, $l \underset{\text{mod}}{\otimes} k \mapsto l \otimes k$ in which $L \underset{\text{mod}}{\otimes} K$ denotes the usual tensor product of F-modules.*

Corollary 1.3. *Let $0 \longrightarrow M \xrightarrow{\delta} L \xrightarrow{\sigma} P \longrightarrow 0$ be a short exact sequence of Lie algebras, K be an arbitrary Lie algebra which acts compatibly on M, L and P and the Lie algebras M, L, P also act compatibly on K. Then the following sequence is exact:*

$$M \otimes K \xrightarrow{\delta \otimes 1_K} L \otimes K \xrightarrow{\sigma \otimes 1_K} P \otimes K \longrightarrow 0.$$

2. On nilpotency and solvability of tensor products

Let L and K be two Lie algebras acting compatibly on each other. We denote by $[L, K]^L$ the submodule of L generated by the elements of the form ${}^k l$ with $l \in L$ and $k \in K$. The submodule $[L, K]^K$ of K is defined similarly. It follows from the compatibility conditions that $[L, K]^L$ and $[L, K]^K$ are ideals of L and K, respectively. Note that when $L = K$ and the actions are the Lie multiplication, we have $[L, K]^L = L^2$, the derived subalgebra of L. In this section, we prove that the non-abelian tensor product $L \otimes K$ is nilpotent, solvable or an Engel Lie algebra if such information is given on one of the above ideals. Also, some bounds for the nilpotency class and solvability length of $L \otimes K$ in terms of the nilpotency class and solvability length of $[L, K]^L$ (or $[L, K]^K$) will be presented.

We first need the following lemma.

Lemma 2.1. (i) *There are epimorphisms* $\mu_L : L \otimes K \to [L, K]^L$, $\mu_K : L \otimes K \to [L, K]^K$ *such that* $\mu_L(l \otimes k) = -{}^k l$, $\mu_K(l \otimes k) = {}^l k$.

(ii) $\ker \mu_L$ *and* $\ker \mu_K$ *are central ideals of* $L \otimes K$.

(iii) *If* $t \in L \otimes K, l' \in L, k' \in K$, *then* $\mu_L(t) \otimes k' = -{}^{k'} t$ *and* $l' \otimes \mu_K(t) = {}^{l'} t$.

The following theorem is analogous to the work of M.P. Visscher [7] in the case of groups, the proof of which is quite different.

Theorem 2.2. *Let L and K be Lie algebras which act on each other in a compatible way.*

(i) *If* $[L, K]^L$ *is nilpotent, then so are* $L \otimes K$ *and* $[L, K]^K$. *Furthermore, if the nilpotency class of* $[L, K]^L$ *is* $\mathrm{cl}([L, K]^L)$, *then*

$$\mathrm{cl}([L, K]^L) \leq \mathrm{cl}(L \otimes K) \leq \mathrm{cl}([L, K]^L) + 1 \text{ and } \mathrm{cl}([L, K]^K) \leq \mathrm{cl}([L, K]^L) + 1.$$

(ii) *If* $[L, K]^L$ *is solvable, then so are* $L \otimes K$ *and* $[L, K]^K$. *Furthermore, if the solvability length of* $[L, K]^L$ *is* $\ell([L, K]^L)$, *then*

$$\ell([L, K]^L) \leq \ell(L \otimes K) \leq \ell([L, K]^L + 1) \text{ and } \ell([L, K]^K) \leq \ell([L, K]^L) + 1.$$

(iii) *If* $[L, K]^L$ *is Engel, then so are* $L \otimes K$ *and* $[L, K]^K$, *Furthermore, if* $[L, K]^L$ *is n-Engel, then* $L \otimes K$ *and* $[L, K]^K$ *are* $(n + 1)$*-Engel.*

Clearly, the above theorem holds when $[L, K]^L$ is replaced by $[L, K]^K$. Thus we obtain bounds for the nilpotency class and the solvability length in terms of both $[L, K]^L$ and $[L, K]^K$.

The following corollary is an immediate consequence of the above theorem.

Corollary 2.3. *Let L be a Lie algebra. Then*

(i) *If the derived subalgebra* L^2 *of* L *is nilpotent of class* $\mathrm{cl}(L^2)$, *then* $L \otimes L$ *is nilpotent of class* $\mathrm{cl}(L^2)$ *or* $\mathrm{cl}(L^2) + 1$.

(ii) *If* L^2 *is solvable of derived length* $\ell(L^2)$, *then* $L \otimes L$ *is solvable of derived length* $\ell(L^2)$ *or* $\ell(L^2) + 1$.

(iii) *If* L^2 *is n-Engel, then* $L \otimes L$ *is* $(n + 1)$*-Engel.*

3. Some bounds for the non-abelian tensor product

We first define ${}_{M_2}M_1 = M_1 \cap (M_2 + Z(M_1 + M_2))$. It is not hard to observe that ${}_{M_2}M_1$ is an ideal of L, and ${}_{M_2}M_1 = M_1$ if $M_1 \subseteq M_2$.
The following lemmas shorten the proofs of our main result.

Lemma 3.1. *Let $\Phi(L)$ denote the Frattini ideal of a d-generator nilpotent Lie algebra L of dimension n. Then the factor Lie algebra $L/\Phi(L)$ is abelian of dimension d. In particular, L is abelian if and only if $d = n$.*

Lemma 3.2. *Let M_1 and M_2 be ideals of a given Lie algebra and M be a central subalgebra of $M_1 + M_2$ which is contained in $[M_1, M_2]$. Then ${}_{M_2}M_1 \otimes M \subseteq M \otimes M_2$.*

Now, we are ready to prove the main result of this section.

Theorem 3.3. *Let M_1 and M_2 be two ideals of a Lie algebra L and that actions arrives from Lie multiplication in L. Let M_i be a nilpotent Lie algebra of dimension n_i with the minimal number of generators d_i for $i = 1, 2$. If $dim\,(\frac{{}_{M_2}M_1}{{}_{M_2}M_1 \cap \Phi(M_1)}) = k$, then*

$$dim\,(M_1 \otimes M_2) \leq n_1 n_2 - (k + n_1 - d_1)(n_2 - d_2).$$

Corollary 3.4. *Let L be a d-generator Lie algebra of dimension n. Then*

$$d^2 \leq dim\,(L \otimes L) \leq nd$$

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