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EXPANSIONS OF FILTERS IN *BL*-ALGEBRAS

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ABSTRACT. In this note, the notions of an expansion of filters and (τ,σ)-primary filters are introduced, and related properties are investigated.

1. INTRODUCTION AND PRELIMINARIES

BL-algebra have been invented by P. Hajek [2] in order to provide an algebraic proof of the completeness theorem of ”Basic Logic” (*BL*, for short) arising from the continuous triangular norms, familiar in the fuzzy Logic framework. Also the notion of filter is defined [2]. In this paper we define the notions of an expansion of filters and (τ,σ)-primary filters. Now, we recollect some definitions which will be used in the following. A *BL*-algebra is a structure $(A,\wedge,\vee,*,\rightarrow,0,1)$ such that:
($BL1$) $(A,\wedge,\vee,0,1)$ is a bounded lattice,
($BL2$) $(A,*,1)$ is a commutative monoid,
($BL3$) $*$ and $\rightarrow$ form an adjoint pair i.e, $c\leq a\rightarrow b$ if and only if $a*c\leq b$ for all $a,b,c\in A$ (residuation),
($BL4$) $a\wedge b=a*(a\rightarrow b)$ (divisibility),
($BL5$) $(a\rightarrow b)\vee(b\rightarrow a)=1$ (prelinearity).
[2,3] In each *BL*-algebra A, the following relations hold for all $x,y,z\in A$,
(1) $x\rightarrow(y\rightarrow z)=y\rightarrow(x\rightarrow z)$,
(2) If $x\leq y$, then $y\rightarrow z\leq x\rightarrow z$ and $z\rightarrow x\leq z\rightarrow y$,
(3) $y\leq(y\rightarrow x)\rightarrow x$,
(4) $y\rightarrow x\leq(z\rightarrow y)\rightarrow(z\rightarrow x)$,
(5) $x\rightarrow y\leq(y\rightarrow z)\rightarrow(x\rightarrow z)$.
[2] A deductive system of a *BL*-algebra A is a subset F containing 1 such that if $x\rightarrow y\in F$ and $x\in F$ imply $y\in F$. Hajek [2] defined a filter of a *BL*-algebra A. Note that a subset F of a *BL*-algebra A is a deductive system of A if and only if F is a filter of A [3]. Filter P is prime if and only if $x\vee y\in P$ implies that $x\in P$ or $y\in P$. Let F be a filter of A. Define the set of double complemented elements, $D(F)$, by $D(F)=\{x\in A\mid x^{--}\in F\}$. Let F, G be filters of A. Then $<F\cup G>=\{a\in A: b*c\leq a$ for some $b\in F, c\in G\}$, [1].

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2. EXPANSIONS OF FILTERS

In the sequel let X be a BL-algebra.

Definition 2.1. *Let $F(X)$ be The set of all filters of X. An expansion of filters in X is defined to be a function $\sigma : F(X) \longrightarrow F(X)$ such that*
$(f1)$ $(\forall F \in F(X))$ $(F \subseteq \sigma(F))$.
$(f2)$ $(\forall F, G \in F(X))$ $(F \subseteq G \Rightarrow \sigma(F) \subseteq \sigma(G))$.

The following examples show the expansions of filter in BL-algebras.

Example 2.2. (1) *The function $\sigma_0 : F(X) \longrightarrow F(X)$ defined by $\sigma_0(F) = F$ is an expansion of filters in X.*
(2) *Let $\sigma : F(X) \longrightarrow F(X)$ be defined by $\sigma(F) = D(F)$. Then by [1] we can see that σ is an expansion of filters in X.*

Definition 2.3. *Let τ and σ be expansions of filters. Then a filter F of X is said to be (τ, σ)-primary if*

$$(\forall a, b \in X)(a \vee b \in F, a \notin \tau(F) \Rightarrow b \in \sigma(F)).$$

Note that a filter F of X is (τ_0, σ_0)-primary if and only if it is a prime filter of X, where $(\tau_0$ and σ_0 are the functions in Example $2.2(1)$.

Theorem 2.4. *Let σ and τ be expansions of filters in X. Then F is (τ, σ)-primary if and only if F is (σ, τ)-primary.*

Note that if $\tau = \sigma_0$, then every (σ, σ_0)-primary is called σ-primary.

Theorem 2.5. *Let τ and σ be expansions of filters in X. Then every prime filter of X is (τ, σ)-primary.*

The following example shows that the converse of the above theorem may not be true.

Examples 2.6. *Let $B = \{0, a, b, c, d, 1\}$. Define $*$ and $\rightarrow$ as follow:*

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

$\rightarrow$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(B, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra and it is clear that $F = \{1\}$ is not prime. We can check that the only filters of X are $\{1\}$ and $\{1, c\}$, then $F(X) = \{\{1\}, \{1, c\}\}$. Define $\tau(F) = \{1, c\}$, for all $F \in F(X)$, and so it is expansion of X. Also $\{1\}$ is a τ-primary, since only $b \vee c = \{1\}$ and $c \in \tau(\{1\})$, but $\{1\}$ is not prime filter.

Lemma 2.7. *Let F be a filter of X. Then F is a prime filter if and only if F is (τ, σ)-primary, for all expansions τ and σ.*

Theorem 2.8. *Let τ be an expansion of filters in X. If $\tau(F)$ is a prime filter then F is a (τ, τ)-primary.*

Corollary 2.9. *Let τ and σ be expansions of filters in X and $F \in F(X)$. If $\tau(F)$ is a prime filter, then $\tau(F)$ is a (τ, σ)-primary.*

Theorem 2.10. *Let τ, σ, ζ and δ are expansions of filters in X such that $\tau \subseteq \zeta$ and $\sigma \subseteq \delta$, then every (τ, σ)-primary filter is (ζ, δ)-primary.*

Corollary 2.11. *Let τ and σ be expansions of filters in X. Then every τ-primary is (τ, σ)-primary.*

Let $y \in X$. The function $\sigma_y : F(X) \longrightarrow F(X)$ defined by

$$(\forall F \in F(X))(\sigma_y(F) = \{x \in X : x \vee y \in F\}).$$

Proposition 2.12. *Let $F, G \in F(X)$. Then*
(1) $\sigma_y(F) \in F(X)$,
(2) $F \subseteq \sigma_y(F)$,
(3) $F \subseteq G \Rightarrow \sigma_y(F) \subseteq \sigma_y(G)$,
(4) $\sigma_y(< F \cup G >) \subseteq \sigma_y(< \sigma_y(F) \cup \sigma_y(G) >)$,
(5) $\sigma_y(F/G) = \sigma_y(F)/G$,
(6) $\sigma_y(\sigma_y(F)) = \sigma_y(F)$,
(7) $\sigma_y(F)$ *is prime filter if and only if* $\sigma_y(F)$ *is* (σ_y, σ_y)*-primary,*
(8) F *is* $\sigma_{x \wedge y}$*-primary if and only if* F *is* σ_y *and* σ_x*-primary.*

Hence σ_y is an expansion of filters in X.

Proposition 2.13. *Let P and F be filters of X such that $F \subseteq P$. If P is prime, then $\sigma_y(F) \subseteq P$, for all $y \in X - P$.*

Theorem 2.14. *A filter P of X is prime if and only if $\sigma_y(P) = P$, for all $y \in X - P$.*

Corollary 2.15. *A filter P of X is prime if and only if P is (σ_y, σ_0)-primary for all $y \in X - P$.*

Theorem 2.16. *Let F be (τ, σ)-primary. Then*

$$(\forall F_1, F_2 \in F(X))(F_1 \vee F_2 \subseteq F, \tau(F_1) \nsubseteq F \Rightarrow \sigma(F_2) \subseteq F.$$

For any filters P and Q of X, the $< P, Q >$ of P and Q is defined to be

$$< P, Q >= \cap_{y \in Q} \sigma_y(P) = \{x \in X : x \vee y \in P, \forall y \in Q\}.$$

Then $< P, Q > \in F(X)$.

Theorem 2.17. *Let P be σ-primary. Then*
(i) *If F is a filter of X which is not contained in $\sigma(F)$, then $< P, F >= P$.*
(ii) *$< P, F >$ is σ-primary.*

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