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IRREDUCIBLE SUBGROUPS OF D^* AND NON-TRIVIALITY OF THE FUNCTOR $G_1(D)$

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Abstract. Let D be an F-central division algebra of degree n. It is proved that if either $n = p^\alpha$, p a prime, and D^* has a non-abelian nilpotent subgroup or $n = pq$, where p and q are primes with $p < q$ such that p does not divide $q - 1$ and D^* contains an irreducible non-abelian soluble-by-finite subgroup, then $G_1(D) \neq 1$.

1. Introduction

Given an F-central division algebra D of index n, denote by D' the commutator subgroup of the multiplicative group D^*. We know that the group $G_1(D) = D^*/\mathrm{Nr}(D^*)D'$, where $\mathrm{Nr}(D^*)$ is the image of D^* under the reduced norm of D to F, is an abelian torsion group of bounded exponent dividing n. This group in not trivial in general. For example, if D is the algebra of real quaternions, then $G_1(D) = 1$ whereas for the rational quaternions $G_1(D)$ is isomorphic to a direct product of copies of $\mathbb{Z}_2$, as it is easily checked. Assume that $G_1(D)$ is not trivial, then by Prũfer-Baer Theorem, we conclude that $G_1(D)$ is isomorphic a direct product of $\mathbb{Z}_{r_i}$, where r_i divides n. In this way one may obtain normal maximal subgroups in D^*. But, the question whether D^* contains a maximal subgroup is still open. The structure of the group $G_1(D)$ is investigated recently in various articles. For example in [1], T. Keshavarzipour and M. Mahdavi-Hezavehi showed that if D is a cyclic division algebra with trivial $G_1(D)$, then D is a quaternion division algebra. Also they proved that if either D is a p-algebra or F has a primitive p-th root of unity for some odd prime p dividing $\deg(D)$, then $G_1(D)$ is non-trivial. Now, let D be an F-central division algebra of index p, p a prime. In [4], it is shown that if D^* contains a non-abelian soluble subgroup, then D is a cyclic division algebra. In this case, if $G_1(D)$ is trivial, using Theorem 1 of [1], one concludes that D is an ordinary quaternion division algebra. In other words, if p is odd and D^* contains a non-abelain soluble subgroup, then $G_1(D)$ is non-trivial.

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Thus, for an arbitrary F-central division algebra D, it is natural to ask what subgroups of D^* can result the non-triviality of $G_1(D)$? In the present note we concentrate on the case where $\deg(D) = p^\alpha$ for an odd prime p or $\deg(D) = pq$ for prime numbers p and q such that $p < q$ and p does not divide $q-1$. Using the above results, it is shown that if D^* has a non-abelian nilpotent subgroup in the former case and D^* has an irreducible soluble-by-finite subgroup in the latter case, then $G_1(D)$ is non-trivial.

2. Main results

We begin with the following theorem which describes the relation between the nilpotent subgroups of D^* and the triviality of $G_1(D)$.

Theorem 2.1. *Let D be an F-central division algebra of degree p^α, p a prime. If $G_1(D) = 1$, then the following statements are equivalent:*
a) *D is the ordinary quaternion division algebra;*
b) *D^* contains a non-abelian nilpotent subgroup.*

Proof. (a)$\Rightarrow$(b)By assumption $D = F \oplus Fi \oplus Fj \oplus Fij$, where i, j are symbols that satisfy the conditions $i^2 = -1, j^2 = -1$, and $ij = -ji$. Set $\mathcal{Q}_8 = \langle i, j\rangle$. Clearly $\mathcal{Q}_8$ is a non-ableian nilpotent subgroup of D^*.
(b)$\Rightarrow$ (a) Let G be a non-abelian nilpotent subgroup of D^*. Set $D_1 = F[G]$ and suppose that the length of the upper central series of G is equal to t, i.e.,

$$\zeta_0 G \geq \zeta_1 G \geq ... \geq \zeta_{t-1} G \geq \zeta_t G = 1,$$

where $\zeta_0 G = G$ and $\zeta_i G = [G, \zeta_{i-1} G]$ for $1 \leq i \leq t$. Thus, $1 \neq \zeta_{t-1} G \leq G' \cap Z(G) \leq D_1' \cap Z(D_1)$. Take $1 \neq a \in \zeta_{t-1} G$. Since $a \in D_1'$, we have $\mathrm{Nr}_{D_1/Z(D_1)}(a) = 1$. On the other hand, $a \in Z(D_1)$ and hence $\mathrm{Nr}_{D_1/Z(D_1)}(a) = a^{p^\beta}$ for some $\beta \in \mathbb{N}$ (note that $[D_1 : Z(D_1)]$ is a p-power). Therefore, D^* contains an element b of order p, say. If $b \notin F$, then $F(b)/F$ is a cyclic extension of F such that $[F(b) : F]$ divides $p-1$. On the other hand, $F(b) \leq D$ yields $[F(b) : F]$ is a p-power which is a contradiction. So we conclude that $b \in F$ and hence F contains a primitive p-th root of unity. Now, Theorem 3 in [1] implies that D is the ordinary quaternion division algebra. ■

Corollary 2.2. *Let D be an F-central division algebra of degree p^α, p an odd prime. If D^* has a non-abelian nilpotent subgroup, then $G_1(D)$ is non-trivial.*

The next lemma gives us a useful tool to realize maximal Galois subfields of an F-central division algebra in terms of irreducible subgroups containing a self-centralizing normal abelian subgroup. This result in full generality is proved in [2], here we present a simple proof for our setting.

Lemma 2.3. *Let D be an F-central division algebra and G be an irreducible subgroup of D^*, i.e., $D = F[G]$. If G has an abelian normal subgroup A such that $C_G(A) = A$, then D is a crossed product division algebra over a maximal subfield K. Also for this subfield we have $Gal(K/F) \simeq G/A$.*

Proof. Set $K = F[A]$ and suppose that T is a right transversal of A in G. Clearly $D = \sum_{t \in T} tK$. We claim that the elements of T are linearly independent over K. To prove this, let $\sum_{j=1}^m t_j k_j = 0$ be a relation of minimal length with $t_j \in T$, $k_j \in K$ and $t_1 = 1$. If $a \in A$, then $0 = a(\sum_{j=1}^m t_j k_j)a^{-1} = \sum_{j=1}^m t_j [a, t_j] k_j$. Subtracting from $\sum_{j=1}^m t_j k_j$ yields $\sum_{j=2}^m t_j(k_j - [a, t_j]k_j) = 0$. Being a shorter relation, this must be identically zero, whence $k_j - [a, t_j]k_j = 0$ for all $2 \leq j \leq m$, i.e., $[a, t_j] = 1$ for all $2 \leq j \leq m$. That is $t_j \in C_G(A) = A$ which is a contradiction. This proves the claim. Therefore, $D = \bigoplus_{t\in T} tK$ and hence $[D : K] = [G : A]$. Now consider the map σ of G/A to $\mathrm{Gal}(K/F)$, where $\sigma(xA)(k) = x^{-1}kx$ for all $x \in G$ and $k \in K$. This map is an injection from G/A to $\mathrm{Gal}(K/F)$, thus we obtain $[D : K] = [G : A] \leq |\mathrm{Gal}(K/F)| \leq [K : F]$. Therefore, $[D : F] \leq [K : F]^2$. On the other hand, K is a subfield of D and thus $[K : F]^2 \leq [D : F]$. This yields $[D : K] = |\mathrm{Gal}(K/F)| = [K : F]$. Consequently K/F is a maximal subfield of D which is Galois and $G/A = \mathrm{Gal}(K/F)$. ■

Let D be an F-central division algebra of degree n and G an irreducible soluble subgroup of D^*. By Mal'cev theorem, one can easily show that G contains an abelian normal subgroup of finite index and consequently G has a maximal abelian normal subgroup of finite index. If this subgroup is self-centralizing then by lemma 2.3, D is a crossed product division algebra over a maximal subfield. So it is reasonable to ask when this subgroup is self-centralizing? In [3] and [4], it is shown that when $n = p^\alpha$ for a prime number p the answer is positive. This question remained as a conjecture until M. Shirvani gave a counterexample in the last section of [5], that provides us an F-central division algebra of degree 15 with an irreducible subgroup G whose maximal abelian normal subgroup A does not satisfy the condition $C_G(A) = A$. Therefore, in general it is not true that every maximal abelian normal subgroup of an irreducible subgroup G of D^* is a self-centralizing abelian normal subgroup. But the following theorem describes its reason.

Theorem 2.4. *Let D be an F-central division algebra and G be a soluble-by-finite irreducible subgroup of D^*. If $\deg(D) = pq$, where p and q are distinct primes, $p < q$ and p does not divide $q-1$, then we have either F has a primitive root of unity for some prime dividing pq or D is a cyclic division algebra.*

Proof. Suppose that F has no primitive r-th root of unity with r dividing pq. By Mal'cev Theorem, G contains an abelian normal subgroup A, say, of finite index. Take A maximal in G and set $H = C_G(A)$. We claim that $H = A$. Suppose $H \neq A$, by maximality of A in G we conclude that $Z(H) = A$. Thus $H/Z(H)$ is finite and hence H' is finite. Since $H \trianglelefteq G$ we have $H' \trianglelefteq G$. By assumption $H' \ncong SL(2, 5)$ (Since the degree of D is odd) and hence it must be soluble. Let $H_1 = H^{t-1}$, where H^i denotes i-th term of the derived series of H and t is the least integer that $H^t = 1$. Thus H_1 is abelian. Consider the subgroup H_1A which is abelian normal in G. Again, by maximality of A we must have $H_1 \subseteq A$. Now, set $D_1 = F[H]$. since D_1 is an F- algebra contained in D, $[D_1 : L]$ divides $[D : F]$, where $L = Z(D_1)$. Suppose $r = \deg(D_1)$ and a is an arbitrary element of H_1. Since $H_1 \leq A \leq L$ we must have $\mathrm{Nr}_{D_1/L}(a) = a^r$. On the other hand, $H_1 \leq H' \leq D_1'$. Thus $\mathrm{Nr}_{D_1/L}(a) = 1$ and consequently $a^r = 1$, where r divides $\deg(D)$. If $a \neq 1$, then D^* contains an element c, say, of order p or q. By assumption we have $c \notin F$. Thus, $[F(c) : F] > 1$ and it divides pq and hence either p divides $[F(c) : F]$ or q divides $[F(c) : F]$. If $O(c) = p$, then $[F(c) : F]$ divides $p - 1$ which is a contradiction. Consequently $O(c) = q$

and so $[F(c) : F]$ divides $q-1$, hence $p|q-1$ or $q|q-1$ which is impossible. Therefore, $a = 1$. Since a was arbitrary we must have $H_1 = 1$ contrary to $H_1 \neq 1$. Therefore, H is abelian and hence $H = A$. Finally, by lemma 2.3, D must be a crossed product division algebra over a maximal subfield K, say, with $|\mathrm{Gal}(K/F)| = \deg(D) = pq$. It is a classical result, using Sylow's Theorem, that every finite group of this order must be cyclic. Therefore, D is a cyclic division algebra. ■

Corollary 2.5. *Let D be an F-central division algebra and G is an irreducible soluble-by-finite subgroup of D^*. If $\deg(D) = pq$ where p and q are distinct primes, $p < q$ and p does not divide $q-1$, then $G_1(D)$ is non-trivial.*

Proof. By assumption p and q are odd prime numbers, thus the degree of D is odd. By theorem 2.4, either F has a primitive root of unity for some prime dividing pq or D is a cyclic division algebra. If F has a primitive root of unity for some prime dividing pq then by theorem 3 in [1], $G_1(D)$ is non-trivial. Also if D is cyclic then theorem 1 in [1] implies that $G_1(D)$ is non-trivial as desired. ■

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