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MODULES WITH FINITELY MANY PRIME SUBMODULES

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Abstract. Let M be a unitary module over a commutative ring R. We say that M has FMP property if every prime submodule of M contains a finitely many prime submodules of M. In [4] we have given a characterization of free modules with this property. The main aim of this note is to study some properties of rings and modules with FMP property. Also we will generalize Cohen's theorem for the Noetherian rings to the modules with FMP property.

1. Introduction

Throughout this note, all rings are commutative with identity and all modules are unitary. For a submodule N of an R-module M, the set $\{r \in R | \, rM \subseteq N\}$ is denoted by $(N : M)$ and is called colon of N. If N is a proper submodule of M and $rm \in N$, for some $r \in R$ and $m \in M$ implies either $m \in N$ or $r \in (N : M)$, then N is said to be a prime submodule of M. In [2, Corollary 1.4], it is proved that:

Corollary. If R is a Noetherian domain and F is a free R-module such that every prime submodule of F contains only finitely many prime submodules, then for any primary submodule Q of F, rad Q is prime.

This is a motivation for studying the modules M for which every prime submodule of M contains only finitely many prime submodules of M. In [4] we have given a characterization of free modules with this property.

Definition 1.1. We say that an R-module M has FMP property, if every prime submodule of M contains only finitely many prime submodules of M. Also it is said that a ring R has FMP property if R has FMP property as an R-module.

For example every one-dimensional domain has FMP property, since every nonzero prime ideal is maximal.

The aim of this note is to study the rings and modules which have FMP property.

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2. Main results

2.1. **Modules With Finitely Many Prime Submodules.**

Proposition 2.1. *A finitely generated module M has FMP property if and only if every maximal submodule of M contains only finitely many prime submodules of M.*

Proposition 2.2. *Let F be a free (or a faithfully flat or a finitely generated faithful multiplication) R-module. If F has FMP property, then R has FMP property.*

Remark 2.3. Note that the converse of this proposition is true for finitely generated faithful multiplication modules, since every prime submodule of a finitely generated faithful multiplication module is of the form PM for some prime ideal P of R. In general, the converse of the Proposition 2.2, is not correct, although we have a particular case for the converse of it:

The set of all P-prime submodules of an R-module M is denoted by $Spec_P\, M$. Also the set of all prime submodules of M is denoted by $Spec\, M$. The R-module M is said to be *Zariski-bounded* if there exists a positive integer n such that for every prime ideal P of R, $\mid Spec_P\, M \mid < n$. It is easy to see that if M is a *Zariski-bounded* module and R has FMP property, then M also has FMP property. So the converse of Proposition 2.2 is true for *Zariski-bounded* modules. In general, if for each prime ideal P of R, $Spec_P\, M$ is a finite set and R has FMP property, then M has FMP property, as an R-module.

The prime submodule dimension of an R-module M is defined by

$$dim\, M = sup_k \{N_0 \subset N_1 \subset \ldots \subset N_k \mid \text{each}\, N_i \text{ is a prime submodule of}\, M\}.$$

Lemma 2.4. *If M has FMP property, then $dim\, M$ is finite.*

An R-module M is said to be a *serial module* if every two submodules of M are comparable. It is easy to see that the converse of the above Lemma is true for serial modules.

Proposition 2.5. *Let M be an R module. The following are equivalent:*

(i) *M has FMP property;*
(ii) *for each prime ideal P of R, M_P has FMP property;*
(iii) *for each maximal ideal P of R, M_P has FMP property.*

A ring R is said to be an *arithmetical ring*, if for all ideal I, J and K of R we have, $I + (J \cap K) = (I + J) \cap (I + K)$. It is easy to see that, Prüfer domains and Dedekind domains are arithmetical.

Theorem 2.6. *Every finite dimensional arithmetical ring has FMP property.*

Theorem 2.7. *Let R be a Noetherian ring. If there exists a free (or a faithfully flat or a finitely generated faithful multiplication) R-module F with FMP property, then $dim\, R = 1$.*

Corollary 2.8. *If R is a Noetherian ring with FMP property, then $dim\, R = 1$.*

Theorem 2.9. *Let F be a free R-module and $2 < rank_R\, F < \infty$. Then, F has FMP property if and only if $\frac{R}{P}$ is a finite field, for each prime ideal P of R.*

Theorem 2.10. *Let M be a module over a zero-dimensional ring R. Then M has FMP property if and only if for each prime submodule N of M, $\frac{M}{N}$ has FMP property as an R-module.*

2.2. **On Minimal Prime Submodules of Modules with FMP Property.**

In [1], Lu proves the extension of Cohen's theorem to modules, namely a finitely generated module M is Noetherian if and only if every prime submodule of M is finitely generated. In this section we will prove a generalization of Cohen's theorem for modules with FMP property. First we have a few useful results:

Theorem 2.11. *Let M be a serial module. The following statements are equivalent:*

(i) *M has FMP property;*
(ii) *M satisfies $A.C.C$ on prime submodules;*
(iii) *M satisfies $A.C.C$ on radical submodules.*

Theorem 2.12. *Let M be a serial module. If M satisfies FMP property, then each radical submodule is a radical of a finitely generated submodule.*

Corollary 2.13. *Let M be an R-module. If M satisfies $A.C.C$ on radical submodules, then each prime submodule is a radical of a finitely generated submodule.*

Corollary 2.14. *Let M be a serial module. If M satisfies FMP property, then every prime submodule is minimal over a finitely generated submodule.*

The following theorem is particularly a converse of Corollary 2.13.

Theorem 2.15. *Let M be an R-module. If each prime submodule is a radical of a finitely generated submodule of M, then M satisfies $A.C.C$ on prime submodules.*

Corollary 2.16. *Let M be a serial module. If every prime submodule is a radical of a finitely generated submodule, then M satisfies FMP property.*

Proposition 2.17. *Let M be a finitely generated module, which satisfies FMP property. If M is a semi-local module(that is, $Max(M)$ is a finite set), then $Spec(M)$ is a finite set.*

Corollary 2.18. *If M is a semi-local finitely generated module which has FMP property, then $Min(M)$ is a finite set. Also for each proper submodule N of M, the number of minimal prime submodules over N is finite.*

Lemma 2.19. *If M is a semi-local finitely generated module which has FMP property, then every prime submodule is a radical of a finitely generated submodule.*

The following is the the generalization of Cohen's Theorem that we have promised:

Theorem 2.20. *Let M be a finitely generated semi-local module which satisfies FMP property. If rad N is finitely generated for each finitely generated submodule N of M, then M is a Noetherian module.*

References

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