

INTEGRAL CLOSURES AND HOMOLOGICAL DIMENSIONS

REZA NAGHIPOUR

Faculty of Mathematical Sciences
University of Tabriz
51666-16471, Tabriz, Iran
naghipour@ipm.ir

Abstract Throughout this talk, all rings considered will be commutative Noetherian and will have non-zero identity elements. Such a ring will be denoted by R and a typical ideal of R will be denoted by I . Let K be a non-zero finitely generated module over R . The concept of *weakly G_K -perfect ideal* was introduced by Golod in [5]. He showed that, this new ideal has some nice properties. For instance, he proved that for a weakly G_K -perfect ideal I of R , K is a canonical module for R if and only if $\text{Ext}_R^s(R/I, K)$ is a canonical module for R/I , where $s = \text{grade}_K I$.

The concept of integral closure of an ideal I of R was introduced by Northcott and Rees. Let I_a denote the integral closure of I in R , i.e., I_a is the ideal of R consisting of all elements $x \in R$ which satisfy an equation $x^n + r_1 x^{n-1} + \dots + r_n = 0$, where $r_i \in I^i$, $i = 1, 2, \dots, n$. On the other hand, Golod in [5] defined the important notion of a suitable module K and used it to define G_K -dimension, as a refinement of projective dimension, for finitely generated modules. The G_K -dimension of a finitely generated R -module M is the length of the shortest resolution of M by so-called totally K -reflexive modules, motivated by Gerko in [4] has extended this notion to complexes. Also, in [4] Gerko introduced the interesting concept of Cohen-Macaulay dimension which characterizes the Cohen-Macaulay rings. The aim of this paper is to study weakly G_K -perfect ideals and the structure of associated primes to the integral closure of ideals.

We shall say that an ideal I of R is K -proper if $K/IK \neq 0$, and when this is the case, we define the K -height of I (written $\text{ht}_K I$) to be

$$\inf\{\text{ht}_K \mathfrak{p} \mid \mathfrak{p} \in \text{Supp} K \cap V(I)\} (= \inf\{\text{ht}_K \mathfrak{p} \mid \mathfrak{p} \in \text{Ass}_R K/IK\}).$$

For any K -proper ideal I of R , we denote by $\text{grade}_K I$ the maximum length of all K -sequences contained in I . A well known result of Rees shows that $\text{grade}_K I$ is the least integer i such that $\text{Ext}_R^i(R/I, K) \neq 0$. We say that I is *weakly G_K -perfect* if $\text{Ext}_R^i(R/I, K) = 0$ for $i \neq \text{grade}_K I$.

In the first section we give a characterization for canonical modules. Namely, we show that if (R, \mathfrak{m}) is a local ring, and K is a non-zero finitely generated suitable R -module, then \mathfrak{m} is G_K -perfect if and only if K is a canonical module for R .

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The concept of a suitable (or semidualizing) module was introduced by Golod [5]. An R -module K is said to be *suitable* (*semidualizing*) if the natural homomorphism $R \rightarrow \text{Hom}_R(K, K)$ is an isomorphism, and $\text{Ext}_R^i(K, K) = 0$ for all $i > 0$. Every suitable module is a PG -module, which was introduced by Foxby [2] as a common generalization of a projective module and a Gorenstein module. We recall that an R -module G is called a PG -module if $\text{Hom}_R(G, G)$ is projective and $\text{Ext}_R^i(G, G) = 0$ for all $i > 0$.

The notion of Gorenstein dimension (G -dimension) was introduced by Auslander [1] and was deeply studied by him and Bridger. Foxby defined the concept of G -dimension with respect to a PG -module, and Golod [5] used it to define the concept of G -dimension with respect to a suitable module K (short: G_K -dimension) as follows: For any R -module X let X^* (or, when necessary, X_K^*) denote $\text{Hom}_R(X, K)$. We say that X is K -*reflexive* if the natural homomorphism $X \rightarrow X^{**}$ is an isomorphism. Let $G_K(R)$ be the class of all R -modules such that X is K -reflexive and $\text{Ext}_R^i(X, K) = 0 = \text{Ext}_R^i(X^*, K)$ for all $i > 0$. If there exists an exact sequence

$$0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow X \rightarrow 0$$

of R -modules such that $X_i \in G_K(R)$ for each $0 \leq i \leq n-1$ and $X_n \neq 0$, then we say that X has G_K -dimension at most n , and write $G_K\text{-dim}_R X \leq n$. If such integer n does not exist, then we say that X has *infinite* G_K -dimension, and write $G_K\text{-dim}_R X = \infty$. If an R -module X has G_K -dimension at most n but does not have G_K -dimension at most $n-1$, then we say that X has G_K -dimension n , and write $G_K\text{-dim}_R X = n$. We say that M is G_K -*perfect* if $G_K\text{-dim}_R X = \text{grade } X$; such modules was introduced under the name *quasi-perfect* by Foxby [3].

In the second section we study the structure of associated primes to the integral closure of ideals which have finite homological dimension. In fact we show that, if K is a finitely generated suitable R -module, I is a K -proper integrally closed ideal of R such that $G_K\text{-dim}_R I < \infty$ and K satisfies Serre's condition (S_1) or $\text{grade}_K I > 0$, then $K_{\mathfrak{p}}$ is a canonical $R_{\mathfrak{p}}$ -module for every $\mathfrak{p} \in \text{Ass}_R R/I$.

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