

A NOTE ON QUOTIENT HYPER K-ALGEBRAS

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ABSTRACT. In this manuscript, we show that if $H \stackrel{H}{=} A \oplus B$, then H/A is isomorphic to B , where H is a hyper K-algebra, A and B are closed sets.

1. INTRODUCTION

The study of BCK-algebra was initiated by Imai and Iséki in 1966. Borzooei, Zahedi and et.al. in 2000 introduced and studied the concept of hyper BCK-algebra and K-algebra.

Definition 1.1. [2] Let H be a set containing a constant “0” and “ \circ ” be a hyperoperation (i.e. a map $\circ : H \times H \rightarrow \mathcal{P}^*(H)$) on H . Then H is called a hyper K-algebra (hyper BCK-algebra) if it satisfies K1-K5(HK1-HK4).

$$\begin{array}{ll} \text{K1: } (x \circ z) \circ (y \circ z) < x \circ y, & \text{HK1: } (x \circ z) \circ (y \circ z) \ll x \circ y, \\ \text{K2: } (x \circ y) \circ z = (x \circ z) \circ y, & \text{HK2: } (x \circ y) \circ z = (x \circ z) \circ y, \\ \text{K3: } x < x, & \text{HK3: } x \circ H \ll x, \\ \text{K4: } x < y, y < x \text{ then } x = y, & \text{HK4: } x \ll y, y \ll x \text{ then } x = y, \\ \text{K5: } 0 < x, & \end{array}$$

for all $x, y, z \in H$, where $x < y$ ($x \ll y$) means $0 \in x \circ y$. Moreover for any $A, B \subseteq H$, $A < B$ (resp. $A \ll B$) if there exist $a \in A$ and $b \in B$ such that $a < b$ (resp. if for all $a \in A$, there exists $b \in B$ such that $a \ll b$). If $A, B \subseteq H$, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$ of H .

Definition 1.2. [2, 5] Let I be a non-empty subset of H such that $0 \in I$. Then I is said to be a *hyper K-ideal* of H if $x \circ y < I$ and $y \in I$ implies that $x \in I$ for all $x, y \in H$ and it is *closed* if for every $x, y \in H$, $x < y$ and $y \in I$ imply that $x \in I$.

Theorem 1.3. [4] *Any hyper K-ideal of hyper K-algebra H is a closed set.*

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Definition 1.4. Let I be a subset of H and $\phi \neq S \subseteq H$. Then we say that I is an S -absorbing, if from $x \in I$ and $y \in S$, it follows $x \circ y \subseteq I$.

Theorem 1.5. [4] Let H be a hyper BCK-algebra and I be a hyper BCK-ideal of hyper K-algebra H . Then I is an H -absorbing.

Definition 1.6. A hyper K-algebra H is called (P) -decomposable if there exists a non-trivial family $\{A_i\}_{i \in \Lambda}$ of subsets of H with the property P such that: (i) $H \neq A_i$, (ii) $H = \cup_{i \in \Lambda} A_i$, (iii) $A_i \cap A_j = \{0\}$, $i \neq j$. for all $i \in \Lambda$, In this case, we write $H = \oplus_{i \in \Lambda} A_i(P)$ and say that $\{A_i\}_{i \in \Lambda}$ is a (P) -decomposition for H . If each A_i , $i \in \Lambda$, is an S-absorbing set we write $H \stackrel{S}{=} \oplus_{i \in \Lambda} A_i(P)$.

Definition 1.7. [4] Let \sim be an equivalence relation on H and $A, B \subseteq H$. Then (i) $A \sim B$ if and only if there exist $a \in A$ and $b \in B$ such that $a \sim b$. (ii) $A \approx B$ if and only if for all $a \in A$, there exists $b \in B$ such that $a \sim b$, and for all $b \in B$ there exists $a \in A$ such that $a \sim b$. (iii) \sim is called regular if $a \sim b$ implies that $a \circ c \approx b \circ c$ and $c \circ a \approx c \circ b$, for any $a, b, c \in H$. (iv) \sim is called a congruence relation on H if $a \sim b$ and $x \sim y$ then $a \circ x \approx b \circ y$. (v) \sim is called good, if $a \circ b \sim \{0\}$ and $b \circ a \sim \{0\}$ implies $a \sim b$ for any, $a, b, c \in H$.

Note: Henceforth, H is a hyper K-algebra and we call an equivalence, regular and good relation by “ERG relation”, the equivalence class x by C_x and $I = C_0$.

Proposition 1.8. [4] If \sim is an ERG relation on H , then $(H/\sim, *, C_0)$ is a hyper K-algebra, where $C_x * C_y = \{C_t | t \in x \circ y\}$

Theorem 1.9. [4] (First isomorphism theorem) Let $f : H_1 \rightarrow H_2$ be a homomorphism, i.e. f is a map such that, $f(0) = 0$ and $f(x \circ y) = f(x) \circ f(y)$, for all $x, y \in H_1$. Then $H_1/Ker f \cong Im(f)$, where $Ker f = \{x \in H_1 : f(x) = 0\}$.

2. MAIN RESULTS

In this section we investigate the characteristics of an ERG relation, \sim , on a hyper K-algebra. We show that any such relation is as form: $x \sim y \Rightarrow x \circ y < I$ and $y \circ x < I$, where $I = C_0$. Moreover if $H \stackrel{H}{=} \oplus_{i \in \Lambda} A_i(\text{closed})$, then this relation change into $x, y \in I$ or $x = y \notin I$ and under some conditions this relation is unique.

Theorem 2.1. (i) Let \sim be an ERG relation on H . Then $x \sim y$ if and only if $x \circ y < I$ and $y \circ x < I$. (ii) Let $H \stackrel{H}{=} A \oplus B(\text{closed set})$. Then $a \circ b < A$ and $b \circ a < A$ if and only if $a, b \in A$ or $a = b \notin A$.

Theorem 2.2. [3] Let $H \stackrel{H}{=} A \oplus B(\text{closed set})$, and $u \sim v$ if and only if $u, v \in A$ or $u = v \notin A$. Then (i) \sim is an equivalence relation on H such that $C_0 = A$

and $|H/\sim| = |B|$. (ii) \sim is a good relation. (iii) \sim is a congruence(regular) relation if and only if $b \circ x = b \circ y$ where $x, y \in A$ and $b \notin A$. (iv) if $b \circ x = b \circ y$ where $x, y \in A$, and $b \notin A$, then \sim is the only ERG relation on H such that $C_0 = A$, and $H/A \cong B$.

Consider the following hyper K-algebra:

\circ	0	1	2	3	4	5
0	{0}	{0}	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}	{1}	{0}
2	{2}	{2}	{0,1,2}	{2}	{2}	{0}
3	{3,4}	{3,4}	{3,4}	{0,3,4}	{0,4}	{3}
4	{4}	{4}	{4}	{4}	{0,4}	{4}
5	{5}	{5}	{5}	{5}	{5}	{0,5}

It is clear that $H = A \oplus B$ (hyper K-ideal) where $A = \{0, 1, 2\}$ and $B = \{0, 3, 4, 5\}$, are H-absorbing and $b \circ x = b \circ y$ for all $b \notin A$ and $x, y \in A$. Let $\sim = \{(0, 0), (1, 0), (0, 1), (1, 1), (0, 2), (2, 0), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$. Then by Theorem 2.2, \sim is an ERG relation on H . Moreover $(H/\sim, C_0, *)$, is as follows:

$*$	C_0	C_3	C_4	C_5
C_0	{ C_0 }	{ C_0 }	{ C_0 }	{ C_0 }
C_3	{ C_3, C_4 }	{ C_0, C_3, C_4 }	{ C_0, C_4 }	{ C_3 }
C_4	{ C_4 }	{ C_4 }	{ C_0, C_4 }	{ C_4 }
C_4	{ C_5 }	{ C_5 }	{ C_5 }	{ C_0, C_5 }

If $H = A \oplus B$ (hyper K-ideal) and at least one of A or B are not H-absorbing, then there is an ERG relation different from the relation given in Theorem 2.2, on H . To this end, consider the following hyper K-algebra on $H = \{0, 1, 2, 3\}$ and ERG relation $\chi = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$ on it. Then $H = A \oplus B$, where $A = \{0, 1\}$ and $B = \{0, 2, 3\}$. Moreover B is not H-absorbing. Also the cayley table of $H/A = \{C_0, C_2\}$ is as follows:

\circ	0	1	2	3	$*$	C_0	C_2
0	{0}	{0}	{0}	{0}	C_0	{ C_0 }	{ C_0 }
1	{1}	{0,1}	{1}	{1}	C_2	{ C_2 }	{ C_0, C_2 }
2	{2}	{2,3}	{0,2}	{1,2}			
3	{3}	{3}	{0,3}	{0,2,3}			

Now we provide an example to show that Theorem 4.1 and Corollary 4.3 of [1], are not true in general.

Remark 2.3. Theorem 4.1. and Corollary 4.3 of [1], are as follows:

“Theorem 4.1: Let H (hyper BCK-algebra) be decomposable with decomposition $H = A \oplus B$. Then there exist a regular(i.e., it is a good relation according to

the our definition) congruence relation Θ on H and a hyper BCK-algebra X of order 2 such that $H/\Theta \cong X$ ”.

“ Corollary 4.3: Let H be decomposable with decomposition $H = A \oplus B$ and $b \circ x = b \circ y$ for all $b \in B$ and $x, y \in A$. Then $|B| = 2$.”

In the proof of Theorem 4.1 of [1], they have claimed that the relation $x\Theta y \iff x, y \in A$ or $x, y \in B - \{0\}$ is a congruence relation and also in Corollary 4.3 they have claimed that $|B| = 2$, but now, we show that these are not true in general. To see this, consider the following example.

Example 2.4. Consider the following hyper BCK-algebra:

0	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{2}	{0,2}	{2}
3	{3}	{3}	{3}	{0,3}

It is easy to check $H = \{0, 1\} \oplus \{0, 2, 3\}$, where $A = \{0, 1\}$ and $B = \{0, 2, 3\}$. By definition of Θ , we have $2\Theta 2$, $3\Theta 2$ but $2 \circ 3 = \{2\} \not\approx 2 \circ 2 = \{0, 2\}$, since $\{0, 2\} \not\subset A$ and $\{0, 2\} \not\subset B - \{0\}$, i.e., Θ is not a congruence relation. Also we see that $|B| \neq 2$.

Since any hyper BCK-ideal is H-absorbing, so if $H = A \oplus B$ (hyper BCK-ideal), then we have $H \stackrel{H}{=} A \oplus B$ (hyper BCK-ideal). Therefore by considering Theorem 2.2, the only ERG relation on H with $C_0 = A$ is exactly as relation: $u, v \in A$ or $u = v \notin A$, hence $x\Theta y \iff x, y \in A$ or $x, y \in B - \{0\}$ is not a congruence relation. The correct version of Corollary 4.3. of [1] is as follows.

Theorem 2.5. Let H be a BCK-algebra and $H = A \oplus B$ (hyper BCK-ideal). Then $H/A \cong B$.

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