

ON δ -SUPPLEMENTED MODULES

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ABSTRACT. In this talk some characterizations of δ -supplemented and δ -lifting modules are given and are investigated some properties of these modules.

1. INTRODUCTION

Throughout this article, all rings are associative with identity, and all modules are unitary right R -modules. A submodule L of a module M is called *small* in M (denoted by $L \ll M$), if for every proper submodule K of M , $L + K \neq M$. A module M is called *hollow*, if every proper submodule of M is small in M .

For two submodules N and K of M , N is called a *supplement* of K in M if N is minimal with the property $M = K + N$; equivalently $M = K + N$ and $N \cap L \ll N$. A module M is called *supplemented* if every submodule of M has a *supplement* in M . Also M is called *amply supplemented* if, for any two submodules L, K of M with $M = L + K$, there exists a *supplement* P of L such that $P \leq K$. A module M is called *weakly supplemented* if, for each submodule A of M , there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll M$. In this case B is called a *weak supplement* of A in M .

Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then B is called a *cosmall* submodule of A in M . A submodule A of M is called *coclosed* in M if A has no proper cosmall submodule. Also B is called a *coclosure* of A in M if B is a cosmall submodule of A and B is coclosed in M .

A submodule K of M is called *essential* in M (denoted by $K \leq^{ess} M$) if $K \cap X \neq 0$ for every non zero submodule X of M . We denote by $Rad(M)$ the radical of M and $R-MOD$ the category of all R -modules. Also we write $A \leq^m M$ to indicate that A is a maximal submodule of M . The *singular submodule* of a module M (denoted by $Z(M)$) is $Z(M) = \{x \in M \mid Ix = 0 \text{ for some ideal } I \leq^{ess} R\}$. A module M is called *singular (nonsingular)* if $Z(M) = M$ ($Z(M) = 0$).

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Let M be a module. A submodule N of M is said to be δ -small in M (notation $N \ll_{\delta} M$) if, whenever $N + X = M$ with M/X singular, $X = M$. The concept of δ -small submodules was introduced by Zhou in [4]. A module M is called δ -hollow, if every proper submodule of M is δ -small in M .

Every small submodule of M is δ -small in M and the converse is true whenever M is singular. But as we see in the next example the converse need not be true in general.

Example 1.1. Let R be a right semisimple ring and M be a nonzero right R -module. Then M is nonsingular and semisimple. For any nonzero $N \leq M$, N is a direct summand of M and hence is not small in M ; but every submodule of M (even M itself) is δ -small in M .

Let N and L be submodules of a module M . N is called a (weak) δ -supplement of L in M , if $N + L = M$ and $N \cap L \ll_{\delta} N$ ($N \cap L \ll_{\delta} M$). A module M is called (weakly) δ -supplemented if every submodule of M has a (weak) δ -supplement in M . M is called *amply* δ -supplemented if, for any submodules A and B of M with $M = A + B$, A has a δ -supplement contained in B .

2. MAIN RESULTS

Lemma 2.1. *Let N and L be submodules of a module M . Then the following are equivalent.*

- (1) N is a δ -supplement of L in M ;
- (2) $N + L = M$ and for each $K \leq N$ with $K + L = M$ and N/K singular, $K = N$.

Lemma 2.2. *Let M be a module and $N \leq M$. Consider the following conditions:*

- (1) N is a δ -supplement submodule of M ;
 - (2) N is weak δ -coclosed in M ;
 - (3) For all $x \leq M$, $x \ll_{\delta} M$ implies $x \ll_{\delta} N$.
- Then (1) \Rightarrow (2) \Rightarrow (3) hold. If M is weakly δ -supplemented, then (3) \Rightarrow (1) holds.*

Lemma 2.3. *For $K \subseteq L \subseteq M$, the following are equivalent:*

- (1) K is a δ -cosmall submodule of L in M ;
- (2) For any $X \leq M$ with M/X singular, $L + X = M$ if and only if $K + X = M$.

Lemma 2.4. *Let M be a module. Then for any $a \in M$ we have: aR is not δ -small in M , if and only if there exists a maximal submodule C of M with M/C singular and $a \notin C$.*

Definition 2.5. Let \wp be the class of all singular simple modules. For a module M let $\delta(M) = \text{Rej}_M(\wp) = \cap \{N \subseteq M \mid M/N \in \wp\}$ be the reject of \wp in M .

From the definition we immediately have $\delta(M/\delta(M)) = 0$, for any module M .

Proposition 2.6. *Given a module M , each of the following sets is equal to $\delta(M)$.*

- (1) $A_1 = \sum\{A \mid A \ll_\delta M\}$.
- (2) $A_2 = \cap\{B \mid B \leq^m M \text{ with } M/B \text{ singular}\}$.
- (3) $A_3 = \cap\{\ker\phi \mid \phi \in \text{Hom}(M, N) \text{ such that } N \text{ is singular simple}\}$.
- (4) $A_4 = \cap\{\ker\phi \mid \phi \in \text{Hom}(M, N) \text{ such that } N \text{ is singular semisimple}\}$.

Proposition 2.7. *Let U and V be submodules of a module M . Assume that V is a δ -supplement of U in M . Then*

- (1) *If $W + V = M$ for some $W \subseteq U$, then V is a δ -supplement of W in M ,*
- (2) *If $K \ll_\delta M$, then V is a δ -supplement of $U + K$ in M ,*
- (3) *For $K \ll_\delta M$ we have $K \cap V \ll_\delta V$ and so $\delta(V) = V \cap \delta(M)$,*
- (4) *For $L \subseteq U$, $(V + L)/L$ is a δ -supplement of U/L in M/L ,*
- (5) *If $\delta(M) \ll_\delta M$, or $\delta(m) \subseteq U$ and if $p : M \rightarrow M/\delta(M)$ is the canonical projection, then $M/\delta(M) = Up \oplus Vp$.*

Proposition 2.8. *Let M be an amply δ -supplemented module. Then every non δ -small submodule N of M contains a δ -supplement submodule N' such that $N/N' \ll_\delta M/N'$.*

Proposition 2.9. *For a submodule $U \subseteq M$, the following are equivalent.*

- (1) *There is a direct summand X of M with $X \subseteq U$ and $U/X \ll_\delta M/X$.*
- (2) *There is a direct summand $X \subseteq M$ and a submodule Y of M with $X \subseteq U$, $U = X + Y$ and $Y \ll_\delta M$.*
- (3) *There is a decomposition $M = X \oplus X'$ with $X \subseteq U$ and $X' \cap U \ll_\delta X'$.*
- (4) *U has a δ -supplement V in M such that $U \cap V$ is a direct summand in U .*
- (5) *There is an idempotent $e \in \text{End}(M)$ with $Me \subseteq U$ and $U(1 - e) \ll_\delta M(1 - e)$.*

Definition 2.10. A module M is called δ -lifting if, for any $A \leq M$, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A/M_1 \ll_\delta M/M_1$.

For example every δ -hollow module is δ -lifting and it is easy to see that every indecomposable δ -lifting module is δ -hollow.

The next Proposition immediately follows from Proposition 2.9 and also can be found in [2, Lemma 2.3]:

Proposition 2.11. *For a module M the following are equivalent.*

- (1) *M is δ -lifting.*
- (2) *For every submodule N of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2 \ll_\delta M$.*
- (3) *Every submodule N of M can be written as $N = N_1 \oplus N_2$ with N_1 a direct summand of M and $N_2 \ll_\delta M$.*

Corollary 2.12. *Every direct summand of a δ -lifting module is δ -lifting.*

Proposition 2.13. *Let M be a δ -lifting module. Then*

- (1) *Any δ -coclosed submodule of M is a direct summand;*

- (2) M is amply δ -supplemented;
- (3) If $N \subseteq M$ is a fully invariant submodule of M , then M/N is a δ -lifting module.

Proposition 2.14. *Let M be an amply δ -supplemented module such that every δ -supplement submodule of M is a direct summand. Then M is δ -lifting.*

Proposition 2.15. *Let M be a module such that every δ -supplement submodule of M is δ -coclosed in M . Then M is δ -lifting if and only if M is amply δ -supplemented and every δ -supplement submodule is a direct summand.*

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