Tarbiat Moallem University, 20^{th} Seminar on Algebra,
2-3 Ordibehesht, 1388 (Apr. 22-23, 2009) pp 173-175

CONJUGACY ASSOCIATION SCHEMES

A. RAHNAMAI BARGHI

Department of Mathematics, Faculty of Science
K. N. Toosi University of Technology
P.O. Box: 16315-1618, Tehran, Iran. `rahnama@kntu.ac.ir`

Abstract. For a given finite group G we construct a transitive permutation group Γ_G whose related association scheme $\mathrm{Inv}(\Gamma_G)$ is commutative. We investigate some relationship between the group theoretical properties of G and the combinatorial theoretical properties of $\mathrm{Inv}(\Gamma_G)$. Furthermore, the character table of the association scheme $\mathrm{Inv}(\Gamma_G)$ is computed.

1. Introduction

Through this talk G is a finite group and $\mathrm{Inn}(G)$ is the inner automorphism group of G. Let $\Gamma = G \rtimes \mathrm{Inn}(G)$ be semidirect product of G by $\mathrm{Inn}(G)$. Hence Γ is a group which its multiplication operation defined by

$$(x,\varphi)(x',\varphi') = (x{x'}^{\varphi^{-1}}, \varphi\varphi'),\ \forall x, x' \in G,\ \varphi, \varphi' \in \mathrm{Inn}(G)$$

Define the action of Γ on G by $g^{(x,\varphi)} = (gx)^{\varphi}$ for all $g \in G$ and $(x,\varphi) \in \Gamma$. It is easy to see that this action gives us a faithful permutation representation on G. Define the image of Γ under the permutation representation by Γ_G. Obviously, $\Gamma_G \leq \mathrm{Sym}(G)$ is transitive and $G_{right} \trianglelefteq \Gamma_G$, where $G_{right} = \{g_{right} : x \mapsto xg\}$, by identifying $g_{right} \sim (g, id)$. Let $\Gamma_e := \{(e,\varphi) : \varphi \in \mathrm{Inn}(G)\}$. So Γ_e is a subgroup of Γ_G which is isomorphic to $\mathrm{Inn}(G)$. Moreover, the orbit of Γ_e containing x is equal to the conjugacy of x in G, i.e. $x^{\Gamma_e} = cl(x)$.

Let R be an arbitrary basis relation in $\mathrm{Inv}(\Gamma_G)$. Since Γ_G is transitive, for each $(x,y) \in R$ there is $y' \in G$ such that $(e,y') \in R$. It follows that $R = \{(g, yg) : g \in G, y \in cl(x_0)\}$ for some $x_0 \in G$. But since $(e,x_0) \in R$ it forces that $R = (e,x_0)^{\Gamma_G}$. In fact, we have

$$\begin{aligned}
(e,x_0)^{\Gamma_G} &= \bigcup\{(e,x_0)^{(g,\varphi_t)} :\ t, g \in G\} \\
&= \bigcup\{(g^t, (x_0 g)^t) :\ t, g \in G\} \\
&= \bigcup_{t\in G} (R_{x_0^{-1}})^{\varphi_t} \\
&= \bigcup_{g \in G} R_{(x_0^{-1})^g} \qquad (1)
\end{aligned}$$

2000 Mathematics Subject Classification: 20C15, 05E30.
keywords and phrases: Association scheme, character.

where $R_{x_0} = \{(g, x_0^{-1}g) : \ g \in G\}$. Therefore, the adjacency matrix $A(R)$ of R by straightforward calculation is equal to $\sum\limits_{g \in cl(x_0)} P_{(g^{-1})_{left}}$, where $x^{g_{left}} = g^{-1}x$. This implies that the adjacency algebra $W(\mathrm{Inv}(\Gamma_G))$ is isomorphic to $Z(\mathbb{C}[G])$ and so $\mathrm{Inv}(\Gamma_G)$ is commutative. Thus the permutation character π of the group Γ_G is multiplicity free.

As we saw in the above, the basis relations of the association scheme $\mathrm{Inv}(\Gamma_G)$ corresponding to the conjugacy classes of the group G, see [1, Example 1.2]. For this reason $\mathrm{Inv}(\Gamma_G)$ is called conjugacy scheme. We refer the reader to [1] and [2] for preliminaries about association schemes.

2. Main results

Theorem 2.1. *The group Γ_G is nilpotent if and only if the conjugacy scheme $\mathrm{Inv}(\Gamma_G)$ is nilpotent.*

Lemma 2.2. *G is a nilpotent group if and only if Γ_G is a nilpotent group.*

Now from Theorem 2.1 and Lemma 2.2 we get the following corollary:

Corollary 2.3. *G is a nilpotent group if and only if the conjugacy scheme $\mathrm{Inv}(\Gamma_G)$ is nilpotent.*

3. Character table of the conjugacy scheme

In this section we calculate characters of the association scheme $\mathrm{Inv}(\Gamma_G)$.

Let G be a finite group and let $g_0 = e, g_1, \ldots, g_d$ be representatives of the conjugacy classes $C_0 = \{e\}, C_1, \ldots, C_d$ of G respectively, with corresponding class sums $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_d$. It is well known that $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_d$ form a basis of the center of the group algebra $Z(\mathbb{C}[G])$. For $i, j, k \in I = \{0, 1, \ldots, d\}$ and $g \in C_k$, the following number

$$a_{ijk} = |\{(x, y) \in G \times G : x \in C_i, y \in C_j, \ xy = g\}|$$

is independent of the choice of g and

$$\mathcal{C}_i\mathcal{C}_j = \sum_{k=0}^{d} a_{ijk}\mathcal{C}_d \tag{2}$$

If $\chi \in \mathrm{Irr}(G)$ and T is a representation affording χ, then for any $z \in Z(\mathbb{C}[G])$ we have $T(z) = \epsilon I$ for some $\epsilon \in \mathbb{C}$. Since T is an algebra homomorphism and the class sums $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_d$ form a basis for the center of the group algebra $\mathbb{C}[G]$, one can define an algebra homomorphism $\omega_\chi : Z(\mathbb{C}[G]) \to \mathbb{C}$ depending on χ which maps $z \in Z(\mathbb{C}[G])$ to ϵ. Calculation of traces in the equation $T(\mathcal{C}_i) = \omega_\chi I$ follows that

$$\chi(1)\omega_\chi(\mathcal{C}_i) = \chi(\mathcal{C}_i) = \sum_{x \in C_i} = |C_i|\chi(g), \ g \in C_i$$

and so $\omega_\chi(\mathcal{C}_i) = \frac{\chi(g)|C_i|}{\chi(1)}$. This equality along with (2) implies that

$$\omega_\chi(\mathcal{C}_i\mathcal{C}_j) = \sum_{k=0}^{d} a_{ijk}\omega_\chi(\mathcal{C}_k) \tag{3}$$

Now we are ready to construct all irreducible characters of the scheme $\mathrm{Inv}(\Gamma_G)$. For any $\chi \in \mathrm{Irr}(G)$ we define the following $\mathbb{C}$-linear transformation

$$\widehat{\omega_\chi} \ : \ W(\mathrm{Inv}(\Gamma_G)) \to \mathbb{C},\ \widehat{\omega_\chi}(A_i) = \omega_\chi(\mathcal{C}_i) \tag{4}$$

where $A_i = \sum_{g\in C_i} P_{(g^{-1})_{left}}$ is a basis element of the adjacency algebra $W(\mathrm{Inv}(\Gamma_G))$. We claim that

$$\mathrm{Irr}(W(\mathrm{Inv}(\Gamma_G))) = \{\widehat{\omega_\chi} : \chi \in \mathrm{Irr}(G)\}.$$

We first show that $\widehat{\omega_\chi}$ is an algebra homomorphism. To do so, we apply $\widehat{\omega_\chi}$ on both sides of the obvious equality:

$$A_iA_j = \sum_{k=0}^{d} a_{ijk}A_k$$

and we get the following equalities:

$$\begin{aligned}
\widehat{\omega_\chi}(A_iA_j) &= \widehat{\omega_\chi}(\sum_{k=0}^{d} a_{ijk}A_k) = \sum_{k=0}^{d} a_{ijk}\widehat{\omega_\chi}(A_k)\\
&= \sum_{k=0}^{d} a_{ijk}\omega_\chi(\mathcal{C}_k) = \omega_\chi(\mathcal{C}_i)\omega_\chi(\mathcal{C}_j) \text{ (by (3) and (4))}\\
&= \widehat{\omega_\chi}(A_i)\widehat{\omega_\chi}(A_j) \text{ (by (4))}.
\end{aligned}$$

as desired. Hence $\widehat{\omega_\chi} \in \mathrm{Irr}(W(\mathrm{Inv}(\Gamma_G))$. On the other hand, all $\widehat{\omega_\chi}$ are distinct, because if $\widehat{\omega_\chi} = \widehat{\omega_\psi}$, for two distinct elements $\chi, \psi \in \mathrm{Irr}(G)$, then $\widehat{\omega_\chi}(A_i) = \widehat{\omega_\chi}(A_i)$, for each A_i. It implies that $\chi(g_i) = \frac{\chi(1)}{\psi(1)}\psi(g_i)$, for each $i = 0, 1, \ldots, d$. This contradicts to the fact that the character table of a group is invertible as a matrix. This proves the above claim. In particular, we have shown that the mapping $\chi \mapsto \widehat{\omega_\chi}$ is a bijection between $\mathrm{Irr}(G)$ and $\mathrm{Irr}(W(\mathrm{Inv}(\Gamma_G)))$.

References

[1] E. Bannai, T. Ito, *Algebraic combinatorics I: association schemes,* Benjamin/Cummings Publishing Company, 1984.

[2] P.-H. Zieschang, *Theory of association schemes*, Springer Monographs in Mathematics, 2005.