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# BANASCHEWSKI'S THEOREM FOR $S$-POSETS: REGULAR INJECTIVITY AND COMPLETENESS

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ABSTRACT. In this talk, the notion of injectivity in the category **Pos-S** of $S$-posets, for a pomonoid $S$, is discussed. First we see that, although there is no non-trivial injective $S$-poset with respect to monomorphisms, **Pos-S** has enough (regular) injectives with respect to regular monomorphisms (sub $S$-posets). Then, recalling Banaschewski's theorem which states that regular injectivity of posets with respect to order-embeddings and completeness are equivalent, we study it for $S$-posets and get some homological classification of pomonoids and pogroups. Among other things, we also see that regular injective $S$-posets are exactly the retracts of cofree $S$-posets over complete posets.

## 1. INTRODUCTION

Banaschewski [1] proves that complete posets are exactly injective posets relative to extremal monomorphisms (order-embeddings), and Sikorski [3] shows the same result for injective Boolean algebras. The main objective of this paper is to study the counterpart of Banaschewski's Theorem for $S$-posets; posets on which the actions of the pomonoid $S$ on them preserve the order.

**Definition 1.1.** A *left poideal* of a pomonoid $S$ is a (possibly empty) subset $I$ of $S$ if it is both a monoid left ideal ($SI \subseteq I$) and a down set ($a \leq b, b \in I$ imply $a \in I$).

**Definition 1.2.** A pomonoid $S$ which has no proper non-empty left poideal is said to be *left simple*.

**Definition 1.3.** By a *complete $S$-poset*, we mean an $S$-poset which is merely complete as a poset.

**Lemma 1.4.** *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors such that $F \dashv G$. Also, let $\mathcal{M}, \mathcal{M}'$ be certain subclasses of $\mathcal{C}, \mathcal{D}$, respectively. If for all $f \in \mathcal{M}$, $Ff \in \mathcal{M}'$, then for any $\mathcal{M}'$-injective object $D \in \mathcal{D}$, $GD$ is an $\mathcal{M}$-injective object of $\mathcal{C}$.*

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## 2. Injectivity and regular injectivity in **Pos-S**

Let **Pos** denote the category of all partially ordered sets (posets) with order preserving (monotone) maps between them. Then we have:

**Lemma 2.1.** **Pos** *has no non-trivial injective object.*

**Theorem 2.2.** **Pos-S** *has no non-trivial injective object.*

**Lemma 2.3.** *Every non-trivial (non-singleton) regular injective S-poset A is bounded by two zero elements.*

**Corollary 2.4.** *Let the identity of the pomonoid S be its top element. If S regarded as an S-poset is regular injective then* $S = \{1\}$.

**Theorem 2.5** (**Pos**-$S$ has enough regular injectives)**.** *Each S-poset can be regularly embedded into a regular injective S-poset.*

**Theorem 2.6.** *An S-poset is regular injective if and only if every regular embedding* $A \to B$ *has a left inverse.*

**Theorem 2.7.** *An S-poset A is regular injective if and only if it is a retract of a cofree S-poset over a complete poset.*

## 3. Regular injectivity and completeness

**Proposition 3.1.** *Every regular injective S-poset is complete.*

Since a regular injective $S$-poset must have two zero elements (Lemma 2.3) and since this is not necessarily the case for complete ones, the converse of the above proposition is not true.

**Theorem 3.2.** *An S-poset A is complete if and only if* $A^{(S)}$ *is a regular injective S-poset.*

**Theorem 3.3.** *Let S be a pomonoid. If all complete S-posets are regular injective, then S is left simple.*

**Theorem 3.4.** *Let S be a commutative pomonoid or a pomonoid whose identity is the top element. Then all complete posets are regular injective as S-posets with trivial actions.*

**Corollary 3.5.** *If the chain* **2** *is a regular injective S-poset then its action is trivial. The converse is true if S is a commutative pomonoid or a pomonoid whose identity is the top element.*

**Lemma 3.6.** *Let S be a pogroup and A be a complete S-poset. Then, for any* $X \subseteq A$ *and* $s \in S$, $(\bigvee X)s = \bigvee Xs$.

**Theorem 3.7.** *Let S be a pogroup. Then an S-poset is complete if and only if it is regular injective.*

**Proposition 3.8.** *Let S be a pomonoid whose identity is a maximal* (*respectively, the top*) *element. Then all complete S-posets are regular injective if and only if S is a pogroup* (*respectively,* $S = \{1\}$).

**Remark 3.9.** We know that if $S$ is a group, any $S$-act is injective if and only if it has a zero element. But, this fact does not hold for regular injectivity in **Pos-S**. For example, taking $S = \{1\}$, and $A$ to be any incomplete $S$-poset, then all elements of $A$ are zero while $A$ is not regular injective.

**Theorem 3.10.** *Let $S$ be a pogroup. Then all complete posets are regular injective as $S$-posets with trivial actions.*

**Remark 3.11.** The converse of the above theorem is not generally true. In fact, if $S$ is a pomonoid and all complete posets are regular injective $S$-posets with trivial actions, then $S$ is not necessarily a pogroup. For example, consider the pomonoid $S = (\mathbb{N}^\infty, min, \leq)$. Since $S$ is commutative, by Theorem 3.4, all complete $S$-posets with trivial actions are regular injective, whereas $S$ is not a pogroup.

## References


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