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# ON THE COMMUTATIVITY DEGREE OF COMPACT GROUPS

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Abstract. In any finite group $G$, the commutativity degree of $G$ (denoted by $d(G)$) is the probability that two randomly chosen elements of $G$ commute. The aim of this paper is to generalize definition of $d(G)$ for every compact group $G$ (infinite and even uncountable). We shall state some results concerning the compact group which are mostly new or improvements of known results given in [4].

## 1. Introduction

Following [1, 5], if $G$ is a finite group, then the *commutativity degree* of $G$, denoted by $d(G)$, is defined as the ratio

$$d(G) = \frac{1}{|G|^2}|\{(x,y) \in G \times G \ \mid \ xy = yx\}|.$$

In fact, $d(G)$ is the probability that a randomly chosen pair of elements of group $G$ commute.

For infinite groups (that most of topological groups are infinite), these ratios are no longer meaningful. In this case, compact groups with normalized Haar measure which are a subclass of topological groups, are a good candidate for this procedure. We know that if $G$ is a group with a locally compact and Hausdorff topology, then $G$ is called a *locally compact topological group*, if the mapping $G \times G \longrightarrow G$, defined by $(x, y) \longmapsto xy^{-1}$ is continuous. It is useful recall that every locally compact topological group $G$ admits a left Haar measure $\mu$, which is a positive Radon measure on a $\sigma$-algebra containing Borel sets with the property that $\mu(xE) = \mu(E)$ for each element $x$ of the measure space $X$ (see [**?**, Sections 18.1 and 18.2]). The support of $\mu$ is $G$ and it is usually unbounded, but if $G$ is compact, then $\mu$ is bounded. For this reason we may assume without ambiguity that a compact group $G$ has a unique probability measure space $(G, \mathcal{M}, \mu)$ with normalized Haar measure $\mu$ (see [**?**, Theorem

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2.8]). Now, let us state the following definition (going back to Gustafson [1]) which is suited for the compact groups. Let $G$ be a compact group with the normalized Haar measure $\mu$. On the product measure space $G\times G$, it is possible to consider the product measure $\mu \times \mu$ which is a probability measure. If $C_2 = \{(x,y) \in G \times G \mid xy = yx\}$ then $C_2 = f^{-1}(1_G)$, where $f : G \times G \to G$ is defined via $f(x,y) = x^{-1}y^{-1}xy$ and $1_G$ denotes the neutral element of $G$. It is clear that $f$ is continuous and so $C_2$ is a compact and measurable subset of $G \times G$. Therefore it is possible to define $d(G) = (\mu \times \mu)(C_2)$.

Obviously if $G$ is finite, then $G$ is a compact group with the discrete topology and so the Haar measure of $G$ is the counting measure.

## 2. Main results

Let $G$ be a compact group, the Borel filed of $G$ is the $\sigma$-algebra generated by all closed subsets of $G$ which is denoted by $\mathcal{B}(G)$. Also the $\sigma$-algebra of $G$ generated by $\mathcal{B}(G)$ and the zero sets (such $A \subseteq B \in \mathcal{B}(G)$ in which $\mu(B) = 0$) is denoted by $\hat{\mathcal{B}}(G)$. Now, let $H$ be an open subgroup of $G$, $\mu_G$ and $\mu_H$ be corresponding Haar measures. Then we can find a relation between $\mu_G$ and $\mu_H$ as the following:

**Lemma 2.1.** *Let $H$ be an open subgroup of a compact group $G$, then for each $B \in \hat{\mathcal{B}}(H)$, $\mu_H(B) = [G : H]\mu_G(B)$.*

For a locally compact group $G$, we denote the space of complex-valued function on $G$ integrable with respect to the Haar measure by $\mathcal{L}^1(G)$.

**Theorem 2.2.** *Let $H$ be a closed normal subgroup of locally compact group $G$ and $f \in \mathcal{L}^1(G)$. If $\lambda$, $\mu$ and $\nu$ are Haar measures on $H$, $G$ and $G/H$ respectively, then*

$$\int_{\frac{G}{H}} \left( \int_H f(xh)d\lambda(x) \right) d\nu(xH) = \int_G f(x)d\mu(x).$$

Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be two measurable spaces. The mapping $X : \Omega_1 \to \Omega_2$ is said to be a measurable transformation from $(\Omega_1, \mathcal{A}_1)$ to $(\Omega_2, \mathcal{A}_2)$ if $X^{-1}(\mathcal{A}_2) \subseteq \mathcal{A}_1$. Now for measurable transformation $X$ from a measure space $(\Omega_1, \mathcal{A}_1, \mu)$ to a measurable space $(\Omega_2, \mathcal{A}_2)$ one can check that the measure $\mu$ induces a measure $\nu$ on $\mathcal{A}_2$ via $\nu\{A\} = \mu\{X^{-1}(A)\}$, for all $A \in \mathcal{A}_2$. This induced measure is denoted by $\mu X^{-1}$. The next theorem might be called the change of variable theorem since it justifies a technique immortalized by integral calculus.

**Theorem 2.3.** *Let $X$ be a measurable transformation from a measure space $(\Omega_1, \mathcal{A}_1, \mu)$ to a measurable space $(\Omega_2, \mathcal{A}_2)$ and let $\nu = \mu X^{-1}$ be the induced measure on $\mathcal{A}_2$. If $f$ is a real $\mathcal{A}_2$-measurable function on $\Omega_2$, then*

$$\int_{A_2} f d\nu = \int_{A_1} f(X)d\mu.$$

Let $\phi$ be an isomorphism from topological group $G_1$ onto topological group $G_2$. It is clear that $\phi$ is a measurable transformation from $G_1$ to $G_2$, so we can state the following technical lemma.

**Lemma 2.4.** *Assume that* $\mu_1$ *and* $\mu_2$ *are two Haar measures on compact groups* $G_1$ *and* $G_2$ *respectively and* $\phi$ *is an isomorphism from* $G_1$ *to* $G_2$*. Then* $\nu = \mu_1\phi^{-1}$ *is a Haar measure on* $G_2$ *and so* $\nu = \mu_2$*.*

As a consequence of our proof we have the following corollary, which comes directly from Theorem 2.3

**Corollary 2.5.** *Let* $\mu_1$ *and* $\mu_2$ *be two Haar measures on compact groups* $G_1$ *and* $G_2$ *respectively and* $\phi$ *is an isomorphism from* $G_1$ *to* $G_2$*. If* $f$ *is a real measurable function on* $G_2$*, then*

$$\int_{G_2} f d\mu_2 = \int_{G_1} (f\phi) d\mu_1.$$

## 3. COMMUTATIVITY DEGREE

Throughout this section, we assume that $G$ is a compact group that its Haar measure $\mu$ is normalized.

**Lemma 3.1.** *Let* $H$ *be a closed subgroup of a compact group* $G$*, then* $\mu(H) = 1/n$ *if* $[G : H] = n < \infty$*, otherwise* $\mu(H) = 0$*.*

**Lemma 3.2.** *Let* $H$ *be a close subgroup of a compact group* $G$*,* $\mu_G$ *and* $\mu_H$ *be the corresponding Haar measures. Then for all* $x \in G$*,* $\mu_G(C_G(x)) \leq \mu_H(C_H(x))$*.*

**Proposition 3.3.** *Let* $G$ *be a compact group and* $H$ *a closed subgroup of* $G$*, then* $\mu(H)^2 d(H) \leq d(G) \leq d(H)$*. Furthermore, if* $G = HZ(G)$ *then* $d(G) = d(H)$*.*

**Lemma 3.4.** *Let* $G$ *be a non-abelian compact* $p$*-group, then* $d(G) \leq (p^2 + p - 1)/p^3$ *and the equality holds if and only if* $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$*.*

**Lemma 3.5.** *Assume that* $H$ *and* $N$ *are closed subgroups of compact group* $G$ *such that* $N$ *lies in* $H$ *and is a normal subgroup of* $G$ *. Then* $\mu(H) = \nu(H/N)$ *where* $\mu$ *and* $\nu$ *are Haar measures on* $G$ *and* $G/N$ *respectively.*

The following result is a generalization of [4, lemma 1.4] which was proved by Lescot [4] in finite case.

**Theorem 3.6.** *Let* $G$ *be a compact group and* $N$ *closed normal subgroup of* $G$*. Then* $d(G) \leq d(N)d(G/N)$ *and the equality holds if* $N \cap G' = 1$*.*

Now, deal with the notion of isoclinism between two groups which is a generalization of isomorphism and was introduced by P.Hall [2] as the following:

**Definition 3.7.** For two groups $G$ and $H$; a pair $(\varphi, \psi)$ is called an *isoclinism* from $G$ to $H$ if
(i) $\varphi$ is an isomorphism from $G/Z(G)$ to $H/Z(H)$;
(ii) $\psi$ is an isomorphism from $G'$ to $H'$ ;
(iii) the following diagram is commutative:

$$\begin{tikzcd}
\frac{G}{Z(G)} \times \frac{G}{Z(G)} \arrow[r, "\varphi"] \arrow[d, "a_G"'] & \frac{H}{Z(H)} \times \frac{H}{Z(H)} \arrow[d, "a_H"'] \\
G' \arrow[r, "\psi"] & H'
\end{tikzcd}$$

*where,* $a_G(g_1Z(G), g_2Z(G)) = [g_1, g_2]$ *and* $a_H(h_1Z(H), h_2Z(H)) = [h_1, h_2]$.

P. Lescot [4, Lemma 2.3] proved that two finite isoclinic group have the same commutativity degree. In the next theorem we will show that this fact is true for all compact groups. It is necessary to attend in this case that $\varphi$ and $\psi$ must be topological isomorphic.

**Theorem 3.8.** *Let* $G$ *and* $H$ *be two isoclinic compact groups, then* $d(G) = d(H)$.

**Theorem 3.9.** *For a compact group* $G$ *the following statements are equivalent,*
(i) $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.
(ii) $G$ *is isoclinic to an extra special p-group of order* $p^3$.
(iii) $G/Z(G)$ *is a p-group and* $d(G) = (p^2 + p - 1)/p^3$.

**Theorem 3.10.** *For every two compact groups* $G$ *and* $H$, $d(G \times H) = d(G)d(H)$.

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