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ON THE p-GROUPS WITH DISTINCT NONLINEAR CHARACTER KERNELS

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ABSTRACT. Distinct nonlinear complex irreducible characters of a group, need not to have distinct kernels. In this paper, we consider finite p-groups with the property that nonlinear complex irreducible characters have distinct kernels. The aim of this note is to characterize p-groups with this property containing at most three nonlinear irreducible complex characters.

1. INTRODUCTION

It is easy to see that each normal subgroup of a group is an intersection of some irreducible (complex) character kernels (See [3] for the preliminary definitions). So, by studying the character kernels of a group we may obtain useful information about the structure of the normal subgroups. For the purposes of this paper we give the following definition.

Definition 1.1. Let G be a p-group. We say that G is a DK-group if distinct nonlinear irreducible characters of G have distinct kernels.

In this talk, all groups are assumed to be finite of prime power order. The notations $\mathrm{cd}(G)$ and $\mathrm{c}(G)$ are used to denote the set of character degrees and the nilpotency class of a group G, respectively. Also the Frattini subgroup of G is denoted by $\Phi(G)$. We define $\mathfrak{K}(G)$ to be the set of nonlinear irreducible character kernels of G. The center of a character χ of G is defined as below:

$$Z(\chi) = \{g \in G : |\chi(g)| = \chi(1)\}$$

$Z(\chi)$ is a normal subgroup of G containing $\ker\chi$ and $Z(\chi)/\ker\chi = Z(G/\ker\chi)$ is cyclic (see [3, Lemma 2.27]).

Definition 1.2. Let G be a p-group. We say that G is an extraspecial group if $|Z(G)| = p$ and $Z(G) = G' = \Phi(G)$. The group G is said to be semiextraspecial if G/M is extraspecial for any maximal subgroup M of $Z(G)$.

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The following theorem is the main result of this paper:

Theorem A. *Let G be a* DK*-group with at most three nonlinear irreducible characters. Then one of the following holds:*

1. *G is a semiextraspecial 2-group of class 2 with $|Z(G)| \leq 4$.*
2. *$Z(G)$ is noncyclic of order 4 and $|G'| = 2$. Also,* $\mathrm{cd}(G) = \left\{1, |G : Z(G)|^{1/2}\right\}$.
3. *G is a group of order 32 of class 3 with $|Z(G)| = 2$.*

Furthermore, groups satisfying 1, 2 or 3 are DK*-groups.*

A finite p-group G with $\mathrm{cd}(G) = \left\{1, |G : Z(G)|^{1/2}\right\}$ can be characterized with pure group theoretic conditions. Indeed we have the following proposition:

Proposition 1.3. [2, Theorem B] *Let G be a p-group. Then* $\mathrm{cd}(G) = \left\{1, |G : Z(G)|^{1/2}\right\}$ *if and only if $|G : Z(G)|$ is a square and each normal subgroup of G is either contained in $Z(G)$ or contains G'.*

The following theorem provides an infinite family of DK-groups:

Theorem B. *Let G be a semiextraspecial* 2*-group. Then G is a* DK*-group.*

2. Preliminary lemmas

Lemma 2.1. *Let G be a group. Then $\bigcap_{K \in \mathfrak{K}(G)} K = 1$. In particular, if $\mathfrak{K}(G)$ has a unique minimal element K, then $K = 1$.*

Lemma 2.2. *Let G be a p-group with a unique nonlinear non-faithful irreducible character. Then $|G| = 16$ and $\mathrm{c}(G) = 3$.*

Lemma 2.3. [1]*Let G be a p-group with distinct nonlinear irreducible character degrees. Then it is an extraspecial 2-group. In particular, it has only one nonlinear irreducible character.*

Lemma 2.4. *Let G be a non-abelian p-group with $\mathfrak{K}(G) = \{K_0, ..., K_n\}$. If $K_0 < ... < K_n$ then $|K_i| = p^i$ for $0 \leq i \leq n$.*

Lemma 2.5. *Let G be a non-abelian p-group. Then all of the nonlinear irreducible characters of G are faithful if and only if $Z(G)$ is cyclic and $|G'| = p$.*

3. Main results

In this section, we give all the preliminaries to prove the main theorem of this paper. First of all, we use GAP [5] to observe that DK-groups of class 3 of order 16 do not exist. Also note that a quotient group of a DK-group is again a DK-group. Also by Lemma 2.3, it is clear that DK-groups are 2-groups. We freely use this facts in this section.

Lemma 3.1. *Let G be a* DK*-group with $\mathfrak{K}(G) = \{K_1, ..., K_n\}$, $n > 1$. Then we can not have $K_1 < ... < K_n$.*

Proposition 3.2. [4, Lemma 5.4] *Let G be a p-group. Then G is a semiextraspecial group of class 2 if and only if $G' = Z(G)$ and* $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$.

Lemma 3.3. *Let G be a* DK*-group with $|\mathfrak{K}(G)| = 2$. Then $Z(G)$ is noncyclic of order 4 and $|G'| = 2$. Furthermore,* $\mathrm{cd}(G) = \left\{1, |G : Z(G)|^{1/2}\right\}$.

Lemma 3.4. *Let G be a* DK*-group with $|\mathfrak{K}(G)| = 3$. If at least one of the members of $\mathfrak{K}(G)$ contain the other, then $|G| = 32$.*

Lemma 3.5. *Let G be a* DK*-group with $|\mathfrak{K}(G)| = 3$. If none of the members of $\mathfrak{K}(G)$ contain the other, then $G' = Z(G)$ and* $\mathrm{cd}(G) = \left\{1, |G : Z(G)|^{1/2}\right\}$.

Using these facts prove the Theorem A. Also Theorem B is a conclusion of Theorem A.

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