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PRINCIPALLY WEAKLY AND WEAKLY COHERENT MONOIDS

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Abstract. We shall call a monoid S *principally weakly (weakly)* left *coherent* if direct products of non-empty families of *principally weakly (weakly)* flat right S-acts are *principally weakly (weakly)* flat. Such monoids have not been studied in general. However, Bulman-Fleming and McDowell proved that a commutative monoid S is *(weakly) coherent* if and only if the act S^I is *weakly flat* for each non-empty set I. In this paper we introduce the notion of *finite (principal) weak flatness* for characterizing *(principally) weakly* left *coherent* monoids. Also we investigate monoids over which direct products of acts transfer an arbitrary flatness property to their components.

1. Introduction

In 1960 left (right) *coherent* rings were characterized by S.U. Chase as rings over which every direct product of flat right (left) modules is flat. Furthermore, he proved that a ring R is left *coherent* if and only if R_R^I is flat for each non-empty set I. Taking inspiration from the assertions, the problem of finding conditions under which various flatness properties of acts over monoids are preserved under direct products has been investigated by some authors. The present paper addresses some versions of the problem. In 1991, Bulman-Fleming reached the target for projectivity. Nearly that time, Victoria Gould solved the corresponding problem for strong flatness and implicitly for conditions (P) and (E) (see Gould (1992, Proposition 5.2)). Indeed, they have proved that for a monoid S, for every family $\{A_i \mid i \in I\}$ of projective (strongly flat, satisfying Condition (P), satisfying Condition (E)) right S-acts, $\prod_I A_i$ is projective (strongly flat, satisfying Condition (P), satisfying Condition (E)) if and only if the act S^I is so, which can be considered suitable counterpart of coherency in ring theory. But for principally weakly and weakly flat acts this is not the case. In this paper we obtain a characterization for *(principally)*

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weakly left *coherent* monoids with the notion of *finite (principal) weak flatness.* Then we investigate on the classification of monoids by flatness, Condition (P), Condition (E), strong flatness, etc. of direct products and their components.

2. Main results

Definition 2.1. A monoid S is called *right finite (principally) weakly flat* if for each $s \in S$, there exists $n_s \in \mathbb{N}$ such that for every (principally) weakly flat act A_S with $as = a's$ $(a, a' \in A_S)$ there exists an S-tossing of length $k \leq n_s$ connecting (a, s) to (a', s) in $A \times Ss$ of the form

$$\begin{array}{rr} & s_1 s = s \\ as_1 = b_1 t_1 & s_2 s = t_1 s \\ b_1 s_2 = b_2 t_2 & s_3 s = t_2 s \\ \cdots & \cdots \\ b_{k-1} s_k = a' t_k & s = t_k s \end{array}$$

with $k \in \mathbb{N}$, $s_1, \ldots, s_k, t_1, \ldots, t_k \in S$ and $b_1, \ldots, b_{k-1} \in A_S$.

Theorem 2.2. *A monoid S is principally weakly left coherent if and only if*

(a) *S_S^I is principally weakly flat for each non-empty set I, and*

(b) *S is right finite principally weakly flat.*

Theorem 2.3. *The following are equivalent for a monoid S:*
(i) *S is weakly left coherent;*

(ii) (a) *the act S^I is weakly flat for each non-empty set I and*

(b) *S is RFWF monoid.*

3. Transferring properties from direct products to their components

This section is devoted to answering the question of when direct products of acts transfer an arbitrary flatness property, projectivity, freeness, and regularity to their components.

Theorem 3.1. *For a monoid S the following conditions are equivalent:*
(i) *for every family $\{A_i \mid i \in I\}$ of right S-acts, if $\prod_I A_i$ is flat, then each A_i is flat;*
(ii) *for every family $\{A_i \mid i \in I\}$ of right S-acts, if $\prod_I A_i$ is weakly flat, then each A_i is weakly flat;*
(iii) *the one-element right S-act Θ_S is (weakly) flat;*

(iv) *S is right reversible;*

(v) *there exists a (weakly) flat right S-act containing a zero.*

Theorem 3.2. *The following assertions are equivalent for a monoid S:*
(i) *if $\prod_I A_i$ satisfies Condition (E) for a family $\{A_i \mid i \in I\}$ of right S-acts, then for each $i \in I$, A_i satisfies Condition (E);*

(ii) *if* $\prod_I A_i$ *is equalizer flat for a family* $\{A_i \mid i \in I\}$ *of right* S*-acts, then for each* $i \in I,\ A_i$ *is equalizer flat;*

(iii) *if* $\prod_I A_i$ *is strongly flat for a family* $\{A_i \mid i \in I\}$ *of right* S*-acts, then for each* $i \in I,\ A_i$ *is strongly flat;*

(iv) Θ_S *is strongly flat;*

(v) S *is left collapsible.*

Concluding this section, we summarize the results with the following table.

Property	**The necessary and sufficient condition on S for transferring the property from direct products to their components**
Principally weak flatness	S needs no condition.
Weak flatness	S is right reversible.
Flatness	S is right reversible.
Condition (P)	S is right reversible.
Equalizer flatness	S is left collapsible.
Condition (E)	S is left collapsible.
Strong flatness	S is left collapsible.
Projectivity	S contains a left zero.
Freeness	$S = \{1\}$.
Regularity (if there exists a regular act)	S contains a left zero.

References

[1] Bulman-Fleming, S. and K. McDowell. (1990). (a). Coherent monoids. in: Latices, Semigroups and Universal Algebra. ed. J. Almeida et al. (Plenum Press, New York).

[2] Bulman-Fleming, S. (1991). Products of projective S-systems. Communications in Algebra. 19(3): 951-964.

[3] Gould, V. (1992). Coherent monoids. J. Austral. Math. Soc. (Series A) 53: 166-182.

[4] Kilp, M., U. Knauer and A. Mikhalev. (2000). *Monoids, Acts and Categories.* W. de gruyter. Berlin.