

CATEGORICAL PROPERTIES OF SEQUENTIALLY DENSE MONOMORPHISMS OF SEMIGROUP ACTS

LEILA SHAHBAZ

Department of Mathematical Sciences
University of Maraghe
Moalleme Shomali Ave. Maraghe, Iran leilashahbaz@yahoo.com
(Joint work with Mojgan Mahmoudi)

ABSTRACT. Let \mathcal{M} be a class of (mono)morphisms in a category \mathcal{A} . To study mathematical notions, such as injectivity, tensor products, flatness, one needs to have some categorical and algebraic information about the pair $(\mathcal{A}, \mathcal{M})$.

In this paper we take \mathcal{A} to be the category **Act-S** of S -acts, for a semigroup S , and \mathcal{M}_d to be the class of sequentially dense monomorphisms (of interests to computer scientists, too) and study the categorical properties, such as limits and colimits, of the pair $(\mathcal{A}, \mathcal{M})$. Injectivity with respect to this class of monomorphisms have been studied by Giuli, Ebrahimi, and the authors and got information about injectivity relative to monomorphisms.

1. INTRODUCTION

Let \mathcal{M} be a class of morphisms of a category \mathcal{A} . To study mathematical notions, such as injectivity and flatness, one needs to have some categorical and algebraic information about the pair $(\mathcal{A}, \mathcal{M})$.

In this paper we take \mathcal{A} to be the category **Act-S** of (right) acts over a semigroup S and \mathcal{M}_d to be the class of sequentially dense monomorphisms, to be defined in section 2, and study the categorical properties of this pair which are usually related to the behaviour of \mathcal{M}_d -injectivity.

In the following we first recall some facts about the category **Act-S** needed in this paper.

Let S be a semigroup and A be a set. If we have a mapping (called the *action* of S on A)

$$\begin{aligned} \mu : A \times S &\rightarrow A \\ (a, s) &\longmapsto as := \mu(a, s) \end{aligned}$$

such that $a(st) = (as)t$ for $a \in A, s, t \in S$, we call A a (*right*) S -act or a (*right*) act over S .

If S is a monoid with its identity 1, we usually also require that $a1 = a$ for $a \in A$.

2000 Mathematics Subject Classification: 08B25, 18A20, 18A30, 20M30, 20M50.
keywords and phrases: Sequential closure, sequential dense.

A *homomorphism* (or an *equivariant map*, or an *S-map*) from an *S-act* A to an *S-act* B is a function from A to B such that for each $a \in A, s \in S$, $f(as) = f(a)s$.

Since the identity maps and the composition of two equivariant maps are equivariant, we have the category **Act-S** of all right *S-acts* and *S-maps* between them.

2. SEQUENTIAL CLOSURE OPERATORS

In this section we introduce a closure operator, where the dense monomorphisms resulting from it is the subject of study in this paper. First note that, denoting the lattice of all subacts of an *S-act* B by $SubB$, we get:

Definition 2.1. The *sequential closure operator* $C^d = (C_B^d)_{B \in \mathbf{Act-S}}$ on **Act-S** is defined as

$$C_B^d(A) = \{b \in B : bS \subseteq A\}$$

for any subact A of an *S-act* B .

Now, one has the usual two classes of monomorphisms related to the notion of a closure operator as follows:

Definition 2.2. Let $A \leq B$ be in **Act-S**. We say that A is *C^d-closed* in B if $C_B^d(A) = A$, and it is *C^d-dense* (or *sequentially dense* or *s-dense*) in B if $C_B^d(A) = B$. Also, an *S-map* $f : A \rightarrow B$ is said to be *C^d-dense* (*C^d-closed*) if $f(A)$ is a *C^d-dense* (*C^d-closed*) subact of B .

We take \mathcal{M}_d to be the set of all *C^d-dense* monomorphisms.

As the following result shows, C^d is not idempotent in general.

Theorem 2.3. C^d is idempotent ($C_B^d(C_B^d(A)) = C_B^d(A)$ for all *S-acts* B and $A \leq B$) if and only if $S^2 = S$.

3. CATEGORICAL PROPERTIES OF s-DENSE MONOMORPHISMS

In this section we study some categorical and algebraic properties of the category **Act-S** with respect to sequentially dense monomorphisms. We study the composition, limit, and colimit properties in the following three subsections.

3.1. Composition properties of s-dense monomorphisms. The class \mathcal{M}_d is clearly isomorphism closed; that is, contains all isomorphisms and is closed under composition with isomorphisms. But, unfortunately \mathcal{M}_d is not always closed under composition:

Lemma 3.1. The class \mathcal{M}_d is closed under composition if and only if the *C^d-closure operator* is idempotent.

Proposition 3.2. The composition of an *s-dense monomorphism* with a *surjective morphism* is an *s-dense morphism*.

Proposition 3.3. *Let $f : A \rightarrow B \in \mathbf{Act-S}$. Then there are unique (always up to isomorphisms) morphisms $e, m \in \mathbf{Act-S}$ such that:*

- (1) (right \mathcal{M}_d -factorization) $f = me$ with $m \in \mathcal{M}_d$, and
- (2) (diagonalization property of the factorization) for any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & D \\ e \downarrow & & \downarrow g \\ C & & E \\ m \downarrow & & \downarrow v \\ B & \xrightarrow{v} & E \end{array}$$

in $\mathbf{Act-S}$ with $g : D \rightarrow E \in \mathcal{M}_d$, there is a uniquely determined morphism $w : C \rightarrow D$ with $gw = vm$ and $we = u$.

3.2. Limits of s -dense monomorphisms.

Proposition 3.4. \mathcal{M}_d is closed under products.

Proposition 3.5. The class of sequentially dense monomorphisms is closed under \mathcal{M}_d -pullbacks.

Proposition 3.6. The class of sequentially dense monomorphisms is stable under \mathcal{M}_d -pullbacks; in the sense that pullback of any s -dense monomorphism along any morphism is again s -dense.

Proposition 3.7. \mathcal{M}_d is closed under limits.

3.3. Colimits of s -dense monomorphisms.

Proposition 3.8. \mathcal{M}_d is closed under coproducts.

Proposition 3.9. \mathcal{M}_d is closed under direct sums.

Proposition 3.10. In $\mathbf{Act-S}$, pushouts transfer s -dense monomorphisms.

Proposition 3.11. The pushout of s -dense monomorphisms belongs to \mathcal{M}_d .

Proposition 3.12. Multiple pushout of s -dense monomorphisms is an s -dense monomorphism. Also, multiple pushouts transfer s -dense monomorphisms.

Definition 3.13. We say that a category \mathcal{A} has \mathcal{M} -bounds if for every set indexed family $\{m_i : A \rightarrow A_i : i \in I\}$ of \mathcal{M} -morphisms there is an \mathcal{M} -morphism $m : A \rightarrow B$ which factors over all m_i 's; that is there are $d_i : A_i \rightarrow B$ with $d_i m_i = m$.

Proposition 3.14. $\mathbf{Act-S}$ has \mathcal{M}_d -bounds.

Definition 3.15. We say that a category \mathcal{A} has \mathcal{M} -amalgamation property, if the morphism m in the definition of \mathcal{M} -bounds factors over all m_i 's through members of \mathcal{M} ; that is d_i 's belong to \mathcal{M} .

Proposition 3.16. $\mathbf{Act-S}$ has \mathcal{M}_d -amalgamation property.

Proposition 3.17. $\mathbf{Act-S}$ has \mathcal{M}_d -directed colimits.

Definition 3.18. We say that a category \mathcal{A} fulfills the \mathcal{M} -chain condition if for every directed system $((A_\alpha)_{\alpha \in I}, (f_{\alpha\beta})_{\alpha \leq \beta \in I})$ whose index set I is a well-ordered chain with the least element 0, and $f_{0\alpha} \in \mathcal{M}$ for all α , there is a (so called “upper bound”) family $(g_\alpha : A_\alpha \rightarrow A)_{\alpha \in I}$ with $h_0 \in \mathcal{M}$ and $g_\beta f_{\alpha\beta} = g_\alpha$.

Proposition 3.19. Act-S fulfills the \mathcal{M}_d -chain condition.

Theorem 3.20. The class of sequentially dense monomorphisms is closed under colimits.

REFERENCES

- [1] B. Banaschewski, *Injectivity and essential extensions in equational classes of algebras*. Queen’s Papers in Pure and Applied Mathematics, **25** (1970), 131-147.
- [2] M. Mahmoudi and L. Shahbaz, *Characterizing semigroups by sequentially dense injective acts*, Semigroup Forum **75** (2007), 116-128.
- [3] M. Mahmoudi and L. Shahbaz, *Well-behaviour of sequential injectivity of acts over semigroups*, to appear in Communications in Algebra.
- [4] M. Mahmoudi and L. Shahbaz, *Sequential dense essential monomorphisms of acts over semigroups*, published online in Applied Categorical Structures.