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# PRESENTATION BY CONJUGATION FOR REDUCED EXTENDED AFFINE WEYL GROUPS

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(Joint work with Saeid Azam)

Abstract. We give several necessary and sufficient conditions for the existence of *the presentation by conjugation* for a reduced extended affine Weyl group. We invent a computational tool by which one can determine simply if a given extended affine Weyl group has such a presentation. As an application, we determine the existence of the presentation by conjugation for a large class of extended affine Weyl groups.

0.1. **Extended affine Weyl groups.** Let $\mathcal{V}$ be a finite dimensional real vector space and $(\cdot,\cdot)$ be a symmetric positive semidefinite bilinear form on $\mathcal{V}$. An element $\alpha \in \mathcal{V}$ is called *non-isotropic* (resp. *isotropic*) if $(\alpha,\alpha) \neq 0$ (resp. $(\alpha,\alpha) = 0$). For a non-isotropic element $\alpha$, we set $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. The set of non-isotropic elements of a subset $A$ will be denoted by $A^\times$.

Throughout this work we assume $(\mathcal{V},(\cdot,\cdot),R)$ is a reduced extended affine root system of rank $\ell$, nullity $\nu$ and twist number $t$ (see [AABGP, Chapter II] for details). We denote the type of $R$ with $X$. So $X = A_\ell$, $B_\ell(\ell \geq 2)$, $C_\ell(\ell \geq 3)$, $D_\ell$, $E_\ell$, $F_4$, or $G_2$. Let $\mathcal{V}^0$ be the radical of the form. By [AABGP, Lemma II.4.15 and Proposition II.4.17], we may find two subspaces $\mathcal{V}^0_1$ and $\mathcal{V}^0_2$ of dimension $t$ and $\nu - t$, respectively, and two semilattices $S_1$ and $S_2$ in $\mathcal{V}^0_1$ and $\mathcal{V}^0_2$, respectively, with $\mathcal{V}^0 = \mathcal{V}^0_1 \oplus \mathcal{V}^0_2$ and

$$(1)\quad R = R(X,S_1,S_2) := (S+S) \cup (\dot{R}_{sh} + S_1 \oplus \langle S_2\rangle) \cup (\dot{R}_{lg} + k\langle S_1\rangle \oplus S_2),$$

where $S := S_1 \oplus \langle S_2\rangle$, and $\dot{R} := \{0\} \cup \dot{R}_{sh} \cup \dot{R}_{lg}$ is an irreducible finite root system of type $X$ and rank $\ell$ with $\dot{R}_{sh}$ as the set of short roots and $\dot{R}_{lg}$ as the set of long roots of $\dot{R}$. Also $k = 3$ if $X = G_2$ and $k = 2$, otherwise. Note that if $t = \nu$, then $\mathcal{V}^0_2 = S_2 = \{0\}$. By [AABGP, Proposition II.4.9], if $X = F_4$ or $G_2$, then $S_1$ and $S_2$ are lattices in $\mathcal{V}^0_1$ and $\mathcal{V}^0_2$, respectively. Also, if $X = B_\ell$ (resp. $X = C_\ell$) with $\ell \geq 3$, then $S_2$ (resp. $S_1$) is a lattice in $\mathcal{V}^0_2$ (resp. $\mathcal{V}^0_1$).

We now recall the definition of an extended affine Weyl group. Let $\dot{\mathcal{V}}$ be the real span of $\dot{R}$ and set $\tilde{\mathcal{V}} := \mathcal{V} \oplus (\mathcal{V}^0)^\star = \dot{\mathcal{V}} \oplus \mathcal{V}^0 \oplus (\mathcal{V}^0)^\star$, where $(\mathcal{V}^0)^*$ is the dual

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space of $\mathcal{V}^0$. Extend the form on $\mathcal{V}$ to $\tilde{\mathcal{V}}$ naturally, by dual pairing, namely

$$(\dot\beta_1 + \delta_1 + \lambda_1, \dot\beta_2 + \delta_2 + \lambda_2) := (\dot\beta_1, \dot\beta_2) + \lambda_1(\delta_2) + \lambda_2(\delta_1), \tag{2}$$

for $\dot\beta_i \in \dot{\mathcal{V}}$, $\delta_i \in \mathcal{V}^0$ and $\lambda_i \in (\mathcal{V}^0)^*$. The *(extended affine) Weyl group* $\mathcal{W}$ of $R$ is by definition the subgroup of $GL(\tilde{\mathcal{V}})$ generated by reflections $w_\alpha$, $\alpha \in R^\times$, defined by $w_\alpha(u) = u - (u, \alpha^\vee)\alpha$, $u \in \tilde{\mathcal{V}}$. We may identify the finite Weyl group $\dot{\mathcal{W}}$ of $\dot R$ as a subgroup of $\mathcal{W}$. We note that the following relations hold in $\mathcal{W}$.

$$w_\alpha^2 = 1, \qquad w w_\alpha w^{-1} = w_{w(\alpha)} \qquad (\alpha \in R^\times, w \in \mathcal{W}). \tag{3}$$

## 1. PRESENTATION BY CONJUGATION

**Definition 1.1.** Let $\hat{\mathcal{W}}$ be the group defined by generators $\hat w_\alpha$, $\alpha \in R^\times$ and relations:

$$\hat w_\alpha^2 = 1, \qquad \hat w_\alpha \hat w_\beta \hat w_\alpha = \hat w_{w_\alpha(\beta)}, \qquad \alpha, \beta \in R^\times.$$

We say that the extended affine Weyl group $\mathcal{W}$ of $R$ has *the presentation by conjugation* if $\mathcal{W} \cong \hat{\mathcal{W}}$.

**Remark 1.2.** Let $\psi : \hat{\mathcal{W}} \longrightarrow \mathcal{W}$ be the group epimorphism induced from the assignment $\hat{\mathcal{W}}_\alpha \mapsto w_\alpha$, $\alpha \in R^\times$ (see (3)). By looking at the literature we see that, $\mathcal{W}$ is said to have the presentation by conjugation if the map $\psi$ is an isomorphism (or equivalently is one-to-one). However, as we will see in Theorem 1.4, $\mathcal{W} \cong \hat{\mathcal{W}}$ if and only if $\psi$ is an isomorphism. We regard this as an important contribution to the theory.

By (3) the assignment $\hat{\mathcal{W}}_\alpha \longmapsto w_\alpha$ induces a unique epimorphism

$$\psi : \hat{\mathcal{W}} \longrightarrow \mathcal{W}. \tag{1}$$

**Definition 1.3.** Let $R$ be a reduced extended affine root system with extended affine Weyl group $\mathcal{W}$. We call $R$ a *minimal* extended affine root system, if there is no $\alpha \in R^\times$ such that the reflections associated to the elements of the set $R^\times \backslash \mathcal{W}\alpha$ generate $\mathcal{W}$.

We now state our main theorem about the presentation by conjugation.

**Theorem 1.4.** *Let $R = R(X, S_1, S_2)$ be a reduced non-simply laced extended affine root system of the form (1) with extended affine Weyl group $\mathcal{W}$. Then the following statements are equivalent:*

(a) *$\mathcal{W}$ has the presentation by conjugation.*
(b) *$Z(\hat{\mathcal{W}}) \cong Z(\mathcal{W})$.*
(c) *$Z(\hat{\mathcal{W}})$ is a free abelian group.*
(d) *The epimorphism $\psi : \hat{\mathcal{W}} \longrightarrow \mathcal{W}$ given by (1) is injective.*
(e) *The trivial collection is the only integral collection for $(S_1, S_2)$.*
(f) *$R$ is a minimal root system.*

**Corollary 1.5.** *If $X_\ell = A_\ell(\ell \geq 2), D_\ell, E_\ell,$ $F_4$ and $G_2$), then $\mathcal{W}$ has the presentation by conjugation.*

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