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A STRUCTURE THEOREM FOR THE AUTOMORPHISM GROUP OF A FINITE 3-GROUP OF MAXIMAL CLASS

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Abstract. Let G be a 3-group of maximal class of order 3^n. In this talk we give a structure theorem for the full automorphism group of G.

1. Introduction

There have been a number of studies of the automorphism groups of p-groups of maximal class (see for example, Baartmans and Woeppel [1]). These concentrate mostly on small automorphism groups. In this talk we consider the full automorphism group of a finite 3-group of maximal class.

Let G be a p-group of maximal class of order p^n and let $\Phi = \Phi(G)$ be the Frattini subgroup of G. It is well-known [4, Satz III. 3.17] that the order of $\mathrm{Aut}^{\Phi}(G)$, the group of all automorphisms of G centralizing G/Φ, divides p^{2n-4}. Moreover the order of $\mathrm{Aut}_p(G)$, the Sylow p-subgroup of the automorphism group of G, divides p^{2n-3}. In this talk we show that for all 3-groups of maximal class and order 3^n, the order of $\mathrm{Aut}^{\Phi}(G)$ is 3^{2n-4}. Also for $p = 3$ we show that $\mathrm{Aut}^{\Phi}(G)$ is a split extension of $\mathrm{Inn}(G)$, the inner automorphism group of G and then we find a complement of $\mathrm{Inn}(G)$ in $\mathrm{Aut}^{\Phi}(G)$. Moreover we give conditions on G for $|\mathrm{Aut}_3(G)| = 3^{2n-3}$. In this case $\mathrm{Aut}_3(G)$ is a split extension of $\mathrm{Aut}^{\Phi}(G)$ by a cyclic group of order 3.

It is straightforward to see that when p is odd, the (full) automorphism group $\mathrm{Aut}(G)$ of G is a split extension of $\mathrm{Aut}_p(G)$ by a subgroup of the direct product of two cyclic groups of order $p-1$, see [1, Section 1]. By using this result we prove a structure theorem for $\mathrm{Aut}(G)$ when $p = 3$.

Throughout this talk the following notation is used. The terms of the lower and the upper central series of G are denoted by $\gamma_i(G)$ and $\zeta_i(G)$, respectively. The centre of G is denoted by $Z = Z(G)$. For $n \geq 4$ we define the 2-step centralizer K_i in G to be the centralizer in G of $\gamma_i(G)/\gamma_{i+2}(G)$ for $2 \leq i \leq n-2$ and define $P_i = P_i(G)$ by $P_0 = G$, $P_1 = K_2$, $P_i = \gamma_i(G)$ for $2 \leq i \leq n$. The degree of commutativity $l = l(G)$ of G is defined to be the maximum integer such that $[P_i, P_j] \leq P_{i+j+l}$ for all $i, j \geq 1$ if P_1 is not abelian and $l = n-3$ if P_1 is abelian.

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Take $s \in G - \bigcup_{i=2}^{n-2} K_i$, $s_1 \in P_1 - P_2$ and $s_i = [s_{i-1}, s]$ for $2 \leq i \leq n-1$. It is easily seen that $\{s, s_1\}$ is a generating set for G and $P_i(G) = \langle s_i, \ldots, s_{n-1} \rangle$ for $1 \leq i \leq n-1$.

If $n = 4$ then obviously G is metabelian since $P_4 = 1$. So $P_1 = \mathcal{C}_G(P_2)$, which implies that $[P_1, P_2] = 1$. Hence P_1 is abelian and so G has positive degree of commutativity $l = 1$. For $n \geq 5$ and $p = 3$, G has degree of commutativity $n-4$ by [2, Theorem 3.13] and so is metabelian. Therefore any 3-group of maximal class of order 3^n, $(n \geq 4)$ is metabelian and has positive degree of commutativity.

2. Main results

In this section we prove a structure theorem for $\mathrm{Aut}(G)$ when G is a 3-group of maximal class of order 3^n. We note that if $n \leq 3$ and G is not cyclic then $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G)$ and $\mathrm{Aut}_3(G) \cong \mathrm{Aut}^{\Phi}(G) \rtimes C_3$. Moreover when G is cyclic then $\mathrm{Aut}_3(G) = \mathrm{Aut}^{\Phi}(G) \cong C_3$. Therefore in the rest of this section we assume that $n \geq 4$.

We deduce the following theorem from Blackburn's observation [2 p.88] which gives us a presentation for G.

Theorem 2.1. *If G is a 3-group of maximal class of order 3^n, then*

$$G \cong \big\langle s, s_1, \ldots, s_{n-1} \big| s_i = [s_{i-1}, s], [s_{n-1}, s] = 1, [s_1, s_2] = s_{n-1}^a \\ s^3 = s_{n-1}^b, s_1^3 s_2^3 s_3 = s_{n-1}^c, s_i^3 s_{i+1}^3 s_{i+2} = 1 \big\rangle,$$

where $a, b, c \in \{0, 1, 2\}$ and $2 \leq i \leq n-1$. For $n > 4$ there exist 3 groups which possess no abelian maximal subgroup given by $c = 0$, $a = 1$ and $b = 0, 1, 2$. If n is even and $n \geq 4$, there exist 4 groups with an abelian maximal subgroup given by $a = b = 0$, $c = 1, 2$ or $a = c = 0$, $b = 0, 1$. If n is odd and $n > 4$ then there exist 3 groups with an abelian maximal subgroup given by $a = b = 0$, $c = 1$ or $a = c = 0$, $b = 0, 1$.

Theorem 2.2. [3, Theorem 3.2] *Let $G = \langle a, b \rangle$ be a two-generated metabelian group. Then the following are equivalent:*

- (i) *For all $u, v \in G'$ there is an automorphism of G that maps a to au and b to bv;*
- (ii) *G is nilpotent.*

By the above theorem we see that if G is a non-cyclic metabelian p-group of maximal class of order p^n, then for any elements $x, y \in G'$ there is an automorphism that maps s to sx and s_1 to $s_1 y$, so $|\mathrm{Aut}^{\Phi}(G)| = p^{2n-4}$. Now we define α_i, $2 \leq i \leq n-1$, by $s^{\alpha_i} = s$ and $s_1^{\alpha_i} = s_1 s_i$. Clearly $[\alpha_i, \alpha_j] = 1$. Also $\alpha_2 = \sigma_s$ has order p.

Lemma 2.3. *Let G be a 3-group of maximal class and order 3^n. Then $|\alpha_i| = |s_i|$ for $i \geq 3$.*

Corollary 2.4. *If G is a 3-group of maximal of order 3^n, then* $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G) \rtimes A$, *where A is an abelian subgroup of* $\mathrm{Aut}(G)$. *Moreover $A \cong C_{3^m} \times C_{3^m}$ when $n = 2m+3$ $(m \geq 1)$ and $A \cong C_{3^m} \times C_{3^{m+1}}$ when $n = 2m+4$ $(m \geq 0)$.*

Now we find a necessary and sufficient condition on G for $|\mathrm{Aut}_3(G) : \mathrm{Aut}^{\Phi}(G)| = 3$. Also we give a structure theorem for $\mathrm{Aut}_3(G)$.

Theorem 2.5. *Suppose that G is a 3-group of maximal class of order 3^n, where $n \geq 4$. Define the map γ by $s^{\gamma} = ss_1$, $s_1^{\gamma} = s_1$ and $s_i^{\gamma} = [s_{i-1}^{\gamma}, s^{\gamma}]$. Then γ extends to an automorphism of G if and only if $s^3 = (ss_1)^3$ and P_1 is abelian.*

Theorem 2.6. *Let G be a 3-group of maximal class and order 3^n. Then $|\mathrm{Aut}_3(G) : \mathrm{Aut}^{\Phi}(G)| = 3$ if and only if there exists an automorphism of G that maps s to ss_1 and s_1 to s_1.*

Corollary 2.7. *Let G be a 3-group of maximal class and order 3^n. Then $|\mathrm{Aut}_3(G) : \mathrm{Aut}^{\Phi}(G)| = 3$ if and only if P_1 is abelian and $s^3 = (ss_1)^3$.*

Theorem 2.8. *Let G be a 3-group of maximal class and order 3^n. If $|\mathrm{Aut}_3(G) : \mathrm{Aut}^{\Phi}(G)| = 3$ then $\mathrm{Aut}_3(G) = \mathrm{Aut}^{\Phi}(G) \rtimes C_3$.*

Now our aim is to find a structure theorem for $\mathrm{Aut}_2(G)$, the Sylow 2-subgroup of $\mathrm{Aut}(G)$. Since $P_1(G)$ and $P_2(G)$ are characteristic subgroups of G , G/P_2 and P_1/P_2 are invariant under $\mathrm{Aut}_2(G)$. So by Maschke's Theorem there exists $s \in G - P_1$ such that $G/P_2 = P_1/P_2 \oplus \langle P_2, s\rangle/P_2$ and $\langle P_2, s\rangle/P_2$ is invariant under $\mathrm{Aut}_2(G)$. In the rest of this section s will be as above. Therefore if $\alpha \in \mathrm{Aut}_2(G)$ then $s^{\alpha} = s^i x$ and $s_1^{\alpha} = s_1^j y$, where $x, y \in P_2$ and $i, j \in \{1, -1\}$.

The following lemma follows at once from Theorem 2.1.

Lemma 2.9. *Let G be a 3-group of maximal class of order 3^n. By considering the presentation of G we define the maps β_j, $j \in \{1, 2, 3\}$ by $s^{\beta_1} = s$, $s_1^{\beta_1} = s_1^{-1}$, $s^{\beta_2} = s^{-1}$, $s_1^{\beta_2} = s_1$, $s^{\beta_3} = s^{-1}$, $s_1^{\beta_3} = s_1^{-1}$. Then*

(i) *β_1 is an automorphism of G if and only if P_1 is abelian and $s^3 = 1$,*
(ii) *β_2 is an automorphism of G if and only if either n is odd and $s_1^3 s_2^3 s_3 = 1$, or n is even, P_1 is abelian and $s^3 = 1$,*
(iii) *β_3 is an automorphism of G if n is even.*

Lemma 2.10. *Let G be a 3-group of maximal class of order 3^n having positive degree of commutativity. If P_1 is not abelian then so is any maximal subgroup of G.*

We deduce the following theorem by the fact that $\mathrm{Aut}(G) \cong \mathrm{Aut}_3(G) \rtimes H$, where $H \leq C_2 \times C_2$.

Theorem 2.11. *Let G be a 3-group of maximal class of order 3^n.*

(i) *If P_1 is not abelian, then $\mathrm{Aut}_2(G) \cong C_2$.*
(ii) *If P_1 is abelian and $(ss_1)^3 = s^3$, then $\mathrm{Aut}_2(G) \cong C_2 \times C_2$ when $|s| = 3$ and $\mathrm{Aut}_2(G) \cong C_2$ when $|s| = 9$.*
(iii) *If P_1 is abelian and $(ss_1)^3 \neq s^3$, then $\mathrm{Aut}_2(G) \cong C_2 \times C_2$ when n is even and $\mathrm{Aut}_2(G) \cong C_2$ when n is odd.*

Lemma 2.12. *Let G be a 3-group of maximal class of order 3^n. Then every element out of P_1 has order 3 or 9. Furthermore when P_1 is abelian, all elements out of P_1 have the same order if and only if $(ss_1)^3 = s^3$.*

The following main theorem is a straightforward consequence of the lemmas above.

Theorem 2.13. *Let G be a 3-group of maximal class of order 3^n. If G has no abelian maximal subgroup, then* $\mathrm{Aut}(G) \cong \mathrm{Aut}^{\Phi}(G) \rtimes C_2$*. If G has an abelian maximal subgroup, then P_1 is abelian and every element out of P_1 has order* 3 *or* 9*.*

(i) *If all elements out of P_1 have order* 3 *then* $\mathrm{Aut}(G) \cong (\mathrm{Aut}^{\Phi}(G) \rtimes C_3) \rtimes (C_2 \times C_2)$ *and if all elements out of P_1 have order* 9 *then* $\mathrm{Aut}(G) \cong (\mathrm{Aut}^{\Phi}(G) \rtimes C_3) \rtimes C_2$*.*

(ii) *Suppose that elements out of P_1 do not have the same order. If n is even then* $\mathrm{Aut}(G) \cong \mathrm{Aut}^{\Phi}(G) \rtimes (C_2 \times C_2)$ *and if n is odd then* $\mathrm{Aut}(G) \cong \mathrm{Aut}^{\Phi}(G) \rtimes C_2$.

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