Some Normal Edge-transitive Cayley Graphs on quaternion Groups[1]

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Abstract

Let G be a group and let S be a subset of G such that $1_G \notin S$ and $S = S^{-1}$. Let $\Gamma = Cay(G, S)$ be a Cayley graph on G relative to S. Then Γ is said to be normal edge-transitive, if $N_{Aut(\Gamma)}(G)$ acts transitively on edges. In this paper we determine all normal edge-transitive Cayley graphs on a quaternion Group Q_{2^n} of valency 4. In addition we give normal edge-transitive Cayley graphs on a quaternion Group Q_{2^n} of some other valencies .

1 Introduction and preliminaries

For a given graph Γ, we denote by $V(\Gamma), E(\Gamma), Aut(\Gamma)$ the vertex set, edge set and automorphism group, respectively. Let G be a group and let S be a subset of G such that $1_G \notin S$ and $S = S^{-1}$. The Cayley graph $\Gamma = Cay(G, S)$ on G relative to S is defined by $V(\Gamma) = G, E(\Gamma) = \{\{g, sg\} | g \in G, s \in S\}$. The graph $\Gamma = Cay(G, S)$ is vertex-transitive, since $Aut(\Gamma)$ contains the right regular representation G. Thus $G \leq Aut(Cay(G, S))$ and this action of G is regular on vertices, that is, G is transitive on vertices and only the identity element of G fixes a vertex. A Cayley graph $\Gamma = Cay(G, S)$ is said to be edge-transitive if $Aut(\Gamma)$ is transitive on edges. In this paper graphs are finite, simple connected and undirected. It is difficult to find the full automorphism group of a graph in general, and so this makes it difficult to decide whether it is edge-transitive. On the other hand we often have sufficient information about the group G to determine $N = N_{Aut(\Gamma)}(G)$, for N is the semidirect product $N = G.Aut(G, S)$, where $Aut(G, S) = \{\sigma \in Aut(G) : S^\sigma = S\}$. Hence we focus attention on those graphs for which $N_{Aut(\Gamma)}(G)$ is transitive on edges. Such a graph is said to be normal edge-transitive. Thus it is often possible to determine whether $Cay(G, S)$ is

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normal edge-transitive. In [?] Praeger gave an approach to analyzing normal edge-transitive Cayley graphs. Houlis in [3] determined the isomorphism types of all connected normal edge-transitive undirected Cayley graphs for Z_{pq} where p, q are primes, and for $G = Z_p \times Z_p, p$ a prime. In [1] we classify the normal edge-transitive Cayley graphs on abelian Groups of valency at most five. In this paper we determine all normal edge-transitive Cayley graphs on a quaternion Group Q_{2^n} of valency 4. In addition we give normal edge-transitive Cayley graphs on a quaternion Group Q_{2^n} of some other valencies .
The group -and graph- theoretic notation and terminology are standard.

The following theorems are basic for Cayley graphs.

Theorem 1.1 *Let $\Gamma = Cay(G, S)$ be a Cayley graph on G relative to S. Then Γ is connected if and only if $G =< S >$.*

Let $\Gamma = Cay(G, S)$ be a Cayley graph on G relative to S, and let $A = Aut(\Gamma)$. Obviously, $A \geq GAut(G, S)$. It is easy to prove the following

Theorem 1.2 *(1) $N_A(G) = GAut(G, S)$.*
(2) $A = GAut(G, S)$ is equivalent to $G \lhd A$.

2 Main results

Throughout this section, let $Q_{2^n} = \langle a, b | b^{2^n} = 1, a^2 = b^{2^{n-1}}, a^{-1}ba = b^{-1} \rangle$ denote the quaternion group of order 2^{n+1}

Theorem 2.1 *[?] Let $\Gamma = Cay(G, S)$ be a Cayley graph on a finite group G. Then Γ is normal edge-transitive if and only if $Aut(G, S)$ is either transitive on S or has two orbits in S which are inverses of each other.*

Lemma 2.2 *Let $\Gamma = Cay(Q_{2^n}, S)$ be a normal edge-transitive Cayley graph on the quaternion group Q_{2^n} of valency 4. Then $Aut(Q_{2^n}, S)$ is transitive on S.*

Lemma 2.3 *Let $\Gamma = Cay(Q_{2^n}, S)$ be a normal edge-transitive Cayley graph on the quaternion group Q_{2^n} such that $|s| > 4$. Then $Aut(Q_{2^n}, S)$ has two orbits T and T^{-1}, that $S = T \cup T^{-1}$.*

Theorem 2.4 *Let $\Gamma = Cay(Q_{2^n}, S)$ be a Cayley graph on the quaternion group Q_{2^n} such that $S = T \cup T^{-1}$, for $S = \{a, ab, ab^{1+l}, ab^{1+l+l^2}, ...ab^{1+l+...+l^{k-2}}\}$, , that $1+l+...+l^{k-1} \equiv 0(mod 2^n)$ and l is a odd. then Γ is normal edge transitive of valency $2k$.*

Theorem 2.5 *Let* $\Gamma = Cay(Q_{2^n}, S)$ *be a Cayley graph on the quaternion group* Q_{2^n} *of valency* 4 . *Then* Γ *is normal edge transitive,if and only if* Γ *be isomorphic with* $\Gamma = Cay(Q_{2^n}, S)$ *where* $S = \{a, ab, ab^{2^{n-1}}, ab^{2^{n-1}+1}\}$.

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