

DUAL STABILIZERS AND DUAL NORMAL BCK-ALGEBRAS

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ABSTRACT. In this note, the notions of dual right and dual left stabilizers of a set in bounded BCK-algebras are defined and the relationship between of them are investigated. Then we define a class of special bounded BCK-algebras called dual normal BCK-algebras. Finally we prove that the dual semisimple bounded BCK-algebras and dual J -semisimple bounded BCK-algebras are all dual normals.

1. INTRODUCTION AND PRELIMINARIES

The study of BCK-algebras was initiated by Y. Imai and K-Iseki [1] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Dual ideals are important in bounded BCK-algebras. In 1986, the notion of dual ideals in bounded BCK-algebras was introduced by J. Meng [4] and gave certain properties of it. Now, in this paper we define the dual left and dual right stabilizers and dual normal BCK-algebras, as mentioned in the abstract.

We give herein the basic notions on BCK-algebras. For further information, we refer to the book [3]. By a BCK-algebra we mean an algebra $(X, *, 0)$ of type $(2,0)$ satisfying the following axioms: for every $x, y, z \in X$,

- (i) $((x * y) * (x * z)) * (z * y) = 0$,
- (ii) $(x * (x * y)) * y = 0$,
- (iii) $x * x = 0$,
- (iv) $x * y = y * x = 0 \Rightarrow x = y$,
- (v) $0 * x = 0$.

We can define a partial ordering \leq by $x \leq y$ if and only if $x * y = 0$. A BCK-algebra X is said to be *commutative* if $x * (x * y) = y * (y * x)$, for all $x, y \in X$.

If there is an element 1 of X satisfying $x \leq 1$, for all $x \in X$, then the element 1 is called *unit* of X . A BCK-algebra with unit is called *bounded*. In a bounded BCK-algebra, we denote $1 * x$ by Nx and $NA = \{Nx \in X \mid x \in A\}$, for all $A \subseteq X$.

A bounded BCK-algebra X is called *involutory* if $NNx = x$, for all $x \in X$.

A nonempty subset D of a bounded BCK-algebra X is called a *dual ideal* if (i) $1 \in D$, (ii) $N(Nx * Ny) \in D$ and $y \in D$ imply that $x \in D$, for any $x, y \in X$.

Given a bounded BCK-algebra X and a nonempty subset A of X . The intersection of all dual ideals containing A is called the dual ideal generated by A , written $[A]$. It is proved that $[A] = \{x \in X : (...(Nx * Na_1) * ...) * Na_n = 0, \text{ for some } a_1, a_2, \dots, a_n \in A\}$. Let A be a nonempty subset of BCK-algebra X . Then the sets $A_i^* = \{x \in X \mid a * x = a, \forall a \in A\}$

and $A_r^* = \{x \in X \mid x * a = x, \forall a \in A\}$ are called the left and right stabilizers of A , respectively and the set $A^* = A_l^* \cap A_r^*$ is called the stabilizer of A .

2. DUAL STABILIZERS

In the sequel let X be a bounded *BCK*-algebra with unit 1.

Definition 2.1. Let A be a nonempty subset of X . Then the sets

$$DA_l = \{x \in X \mid Na * Nx = Na, \forall a \in A\}$$

and

$$DA_r = \{x \in X \mid Nx * Na = Nx, \forall a \in A\}$$

are called the dual left and dual right stabilizers of A , respectively and the set $DA = DA_l \cap DA_r$ is called the dual stabilizer of A .

For convenience the dual stabilizer, dual left and dual right stabilizers of a single element set $A = \{a\}$ are denoted by DS_a , DL_a and DR_a , respectively.

Theorem 2.2. Let X be an involutory *BCK*-algebra and A be a nonempty subset of X . Then:

- (i) $N(DA_l) = (NA)_l^*$, $N(DA_r) = (NA)_r^*$ and $N(DA) = (NA)^*$,
- (ii) $N(A_l^*) = D(NA)_l$, $N(A_r^*) = D(NA)_r$ and $N(A^*) = D(NA)$.

Theorem 2.3. Let A be a nonempty subset of X . Then DA_l is a dual ideal of X .

The following example shows that DA_r is not a dual ideal in general.

Example 2.4. Let $X = \{0, 1, 2, 3, 4\}$ and let $*$ operation be given by the following table

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Then $(X, *, 0)$ is a bounded *BCK*-algebra with unit 4. Consider $A = \{3\}$, it is easy to check that $DA_r = \{1, 4\}$. Now we see that $4 * ((4 * 3) * (4 * 1)) = 4 \in DA_r$ and $1 \in DA_r$ but $3 \notin DA_r$. Hence DA_r is not a dual ideal of X .

Theorem 2.5. Let X be a bounded commutative *BCK*-algebra and A be a nonempty subset of X . Then $DA_r = DA_l = DA$ and so DA_r is a dual ideal of X .

Theorem 2.6. Let A be a nonempty subset of X . Then

- (i) $[A] \cap DA_r = \{1\}$,
- (ii) $DA_r = D[A]_r$,
- (iii) If DA_r is a dual ideal of X , then $DA = DA_r$.

Theorem 2.7. Let A and B be nonempty subsets of X . Then

- (i) $A \cap DA_l = \emptyset$ or $\{1\}$, $A \cap DA_r = \emptyset$ or $\{1\}$ and $A \cap DA = \emptyset$ or $\{1\}$,
- (ii) if $A \subseteq B$, then $DB_l \subseteq DA_l$, $DB_r \subseteq DA_r$ and $DB \subseteq DA$,
- (iii) $A \subseteq D(DA_r)_l \cap D(DA_l)_r$ and $A \subseteq D(DA)$,
- (iv) $DA_l = D(D(DA_l)_r)_l$, $DA_r = D(D(DA_r)_l)_r$ and $DA = D(D(DA))$,

- (v) $D(A \cup B)_l = DA_l \cap DA_l$, $D(A \cup B)_r = DA_r \cap DA_r$ and $D(A \cup B) = DA \cap DB$,
 (vi) $DA_l = \bigcap_{a \in A} DL_a$, $DA_r = \bigcap_{a \in A} DR_a$ and $DA = \bigcap_{a \in A} DS_a$.

Theorem 2.8. *If A is a dual ideal of X , then DA is a dual ideal of X .*

Theorem 2.9. *Let A, B be two dual ideals of X . Then $A \cap B = \{1\}$ if and only if $A \subseteq DB$.*

Theorem 2.10. *Let A be a dual ideal of X . Then $DA_l = DA \subseteq DA_r$. In particular, if DA_r is a dual ideal of X , $DA = DA_l = DA_r$.*

3. DUAL NORMAL BCK-ALGEBRAS

Definition 3.1. A bounded BCK-algebra X is called dual normal BCK-algebra, if the dual right stabilizer DR_a of any element $a \in X$ is a dual ideal of X .

Note that any commutative bounded BCK-algebra is dual normal BCK-algebra.

Theorem 3.2. *The following statements are equivalent:*

- (i) X is dual normal,
 (ii) $DR_a \subseteq DL_a$, $\forall a \in A$,
 (iii) $DR_a = DL_a$, $\forall a \in A$,
 (v) $Nx * Ny = Nx$ implies $Ny * Nx = Ny$, $\forall x, y \in X$.

Definition 3.3. A bounded BCK -algebra X is called dual semisimple if every dual ideal A of X is a sub-summand of X , i.e. there exists dual ideal B of X such that $A \cap B = \{1\}$ and $X = [A \cup B]$.

Theorem 3.4. *Every dual semisimple bounded BCK-algebra is dual normal .*

Definition 3.5. The dual J-radical , denoted by $DJ(X)$, of a bounded BCK-algebra X means the intersection of all maximal dual ideals of X . By Zorn's Lemma the collection of maximal dual ideals of X is nonempty. If $DJ(X) = \{1\}$, then X is called dual J-semisimple.

Theorem 3.6. *Every dual J-semisimple BCK-algebra is dual normal.*

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