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SOME ANNIHILATOR CONDITIONS IN COMMUTATIVE RINGS

A. YOUSEFIAN DARANI

Department of Mathematics
University of Mohaghegh Ardabili
P.O. Box 179, Ardebil, Iran
yousefian@uma.ac.ir and youseffian@gmail.com

ABSTRACT. Let R be a commutative ring and let M be an R-module. Denote by $Z_R(M)$ the set of all zero-divisors of R on M. M is called *strongly primal* (resp. *super primal*) if for arbitrary $a, b \in Z_R(M)$ (resp. every finite subset F of $Z_R(M)$) the annihilator of $\{a, b\}$ (resp. F) in M is non-zero. In this paper we give some results on these classes of modules. Also we provide a relationship among the families of primal, strongly primal and super primal modules.

1. INTRODUCTION

Throughout this paper all rings are commutative with nonzero identity, and all modules are considered to be unitary.

For a given ring R, an R-module M and a submodule N of M, we will denote by $(N :_R M)$ the residual of N by M, the set of all r in R such that $rM \subseteq N$. The annihilator of M, denoted by $ann_R(M)$, is $(0 :_R M)$. For every subset S of R, we denote by $Ann_M(S)$ the set of elements $m \in M$ such that $ma = 0$ for each $a \in S$. An element $r \in R$ is called a zero-divisor on M provided that there exists $0 \neq m \in M$ such that $rm = 0$, that is $Ann_M(r) \neq 0$. We denote by $Z_R(M)$ the set of all zero-divisors of R on M.

An annihilator condition on a commutative ring R is property (A). R is said to have property (A) if every finitely generated ideal I contained in $Z(R)$ has a nonzero annihilator ([1]). A ring with property (A) is called a McCoy ring. An R-module M is said to be McCoy provided that for every finitely generated ideal I of R with $I \subseteq Z_R(M)$, $Ann_M(I) \neq 0$.

Let R be a commutative ring, M an R-module and N a submodule of M. An element $r \in R$ is called prime to N if $rm \in N$ $(m \in M)$ implies that $m \in N$, that is $(N :_M r) = \{m \in M : rm \in N\} = N$. Denote by $S(N)$ the set of all elements of R that are not prime to N. Then N is said to be *primal* if $S(N)$ forms an ideal; this ideal is always a prime ideal, called the adjoint ideal P of N. In this case we also say that N is a P-primal submodule of M . If the zero submodule of M is primal, then M will be called a primal module. It

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is clear that $S(0) = Z_R(M)$. Hence M is a primal R-module if $Z_R(M)$ forms an ideal of R. The ring R is primal if it is primal as an R-module. It is easy to check that a submodule N of an R-module M is *primal* if and only if the factor module M/N is primal as an $R/(N :_R M)$-module. The R-module M is called *strongly primal* (resp. *super primal*) if for arbitrary $a, b \in Z_R(M)$ (resp. every finite subset F of $Z_R(M)$) the annihilator of $\{a, b\}$ (resp. F) in M is non-zero. Clearly, M is a super primal R-module if and only if M is a primal and McCoy.

The submodule N of the R-module M is called *strongly primal* (resp. *super primal*) if M/N is a strongly primal (super primal) $R/(N :_R M)$-module.

It is clear that every super primal module is strongly primal and every strongly primal module is primal. It is shown in Examples 1.1 that the converse implications do not hold.

Example 1.1. (1) *A McCoy R-module need not be strongly McCoy. For example, let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then R is a McCoy ring which is not strongly McCoy.*

(2) *Let $R = \mathbb{Z}$ and consider the R-module $M = \mathbb{Z}_2 \times \mathbb{Z}_3$. Then M is not a primal R-module, while it is McCoy.*

(3) *In this example we use the concept of so-called $A + B$-rings introduced in* [3]. *Let K be a field, w, y and z algebraically independent indeterminates, $M = (w, y, z)K[w, y, z]$, and let $D = K[w, y, z]_M$. Clearly D is a local ring. Let Q be the maximal ideal of D and let P denote the set of height two primes of D. For each $P_\alpha \in \mathrm{P}$, let $Q_\alpha = Q/P_\alpha$. Let $\mathrm{I} = \mathrm{A} \times \mathbb{N}$ where A is an index set for P and let $B = \sum Q_i$ where $Q_i = Q_\alpha$ for each $i = (\alpha, n) \in \mathrm{I}$. Set $R = D + B$ the ring constructed from $D \times B$ by setting (r,a) + (s, b) = (r+s, a+b) and $(r, a)(s, b) = (rs, rb + sa + ab)$. Then $Z(R) = Q + B$ is a prime ideal of R. Hence R is a primal ring which is not McCoy..*

Proposition 1.2. *Let R be a commutative ring, M an R-module and N a submodule of M. Then $f(x) \in R[x]$ is not prime to $N[x]$ if and only if $mf(x) \in N[x]$ for some $m \in M \backslash N$.*

One can easily check that for every module M over a commutative ring R, $Z_R(M) \subseteq Z_{R[x]}(M[x])$ and it is easy to see that if $M[x]$ is a primal $R[x]$-module then $Z_{R[x]}(M[x]) = Z_R(M)[x]$. Combining this result by Proposition 1.2, we get:

Proposition 1.3. *Let R be a commutative ring and let M be an R-module. Then the following conditions are equivalent*

(i) *$M[x]$ is a primal $R[x]$-module;*
(ii) *M is a super primal R-module;*
(iii) *$M[x]$ is a super primal $R[x]$-module;*
(iv) *$M[x]$ is a strongly primal $R[x]$-module.*

Proposition 1.3 shows that if we can find the conditions under which $M[x]$ is a primal $R[x]$-module, then one can have conditions under which the module

M is super primal and hence primal. In the following results we give some conditions for which the module $M[x]$ is primal.

Proposition 1.4. *Let R be a commutative ring, M an R-module and N an irreducible submodule of M. Then*

(1) *$N[x]$ is a primal submodule of $M[x]$.*
(2) *If P is the adjoint ideal of N, then $P[x]$ is the adjoint ideal of $N[x]$.*

Theorem 1.5. *Let R be a commutative ring, M an R-module and N a submodule of M. Then, $N[x]$ is a primal submodule of $M[x]$ if and only if N is a primal submodule of M and M/N is a McCoy $R/(N :_R M)$-module.*

Corollary 1.6. *Let R be a commutative ring and let M be an R-module. Then $M[x]$ is a primal $R[x]$-module if and only if M is a primal R-module and M is a McCoy R-module.*

Proposition 1.7. *Let R be a commutative ring and let M be an R-module. If N is an irreducible submodule of M, then M/N is a McCoy R-module.*

A submodule N of M is called essential if for ever nonzero submodule K of M, $N \cap K \neq 0$. A nonzero submodule N of M is called uniform if every nonzero submodule of M contained in N is essential. So M is uniform if every nonzero submodule of M is essential in M. We are going to prove the main result of the paper. First we need the following Proposition.

Proposition 1.8. *Let R be a commutative ring. Then every uniform R-module is primal.*

Example 1.9. *Assume that $R = \mathbb{Z}$ and consider the $\mathbb{Z}$-module $M = \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $Z_R(M) = 2\mathbb{Z}$, M is a primal R-module. While M is not a uniform R-module. This example shows that the converse of the Proposition 1.8 is not necessarily true.*

Theorem 1.10. *Let R be a commutative ring. If M is a primal R-module of finite Goldie dimension, then M is McCoy.*

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