

## NOTE ON THE EQUALITY OF DECOMPOSABLE SYMMETRIZED TENSORS

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ABSTRACT. We state a necessary and sufficient condition for equality of two nonzero decomposable symmetrized tensors, and also a necessary and sufficient condition for one decomposable symmetrized tensor to be zero, when the symmetrizer is associated with an irreducible character of a Young subgroup of the symmetric group of degree  $m$ .

### 1. INTRODUCTION

Let  $V$  be a complex vector space and let  $m$  be a positive integer. Denote by  $\bigotimes^m V$  the  $m$ th tensor power of  $V$  and by  $x_1 \otimes \cdots \otimes x_m$  the tensor product of the vectors  $x_1, \dots, x_m$ . For each  $\sigma$  in the symmetric group of degree  $m$ ,  $S_m$ , let  $P(\sigma)$  be the unique linear operator on  $\bigotimes^m V$  satisfying

$$P(\sigma)(x_1 \otimes \cdots \otimes x_m) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}.$$

Let  $G$  be a subgroup of  $S_m$  and let  $\lambda$  be an irreducible character of  $G$ . We denote by  $T(G, \lambda)$  the linear operator,

$$T(G, \lambda) = \frac{\lambda(1)}{|G|} \sum_{\sigma \in G} \lambda(\sigma) P(\sigma),$$

where  $|G|$  is the cardinality of  $G$ .

The operator  $T(G, \lambda)$  is a projection, and its range,  $V_\lambda(G)$ , is called the symmetry class of tensors associated with  $G$  and  $\lambda$ . Those elements of  $V_\lambda(G)$  of the form  $T(G, \lambda)x_1 \otimes \cdots \otimes x_m$  are called decomposable symmetrized tensors and are denoted by  $x_1 * \cdots * x_m$  or, briefly, by  $x^*$ .

If  $G = S_m$ , we call  $V_\lambda(G)$  an immanantal symmetry class of tensors, and we denote it by  $V_\lambda$ . A decomposable symmetrized tensor in  $V_\lambda$  is called an immanantal decomposable tensor.

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There are two important problems about symmetry classes of tensors: The "vanishing problem" that asks conditions on the vectors  $x_1, \dots, x_m$  that characterize the vanishing

$$x_1 * \dots * x_m = 0,$$

and the "equality problem" that asks for conditions on the vectors of  $V$  that characterize the equality

$$x_1 * \dots * x_m = y_1 * \dots * y_m,$$

where  $x_1 * \dots * x_m, y_1 * \dots * y_m \neq 0$ .

Indeed, the vanishing problem was solved by Gamas [2] and the equality problem was solved by J.A. Dias da Silva [1] for immanantal symmetry classes.

We will use some definitions and notations, such as partitions and their majorization partial ordering, coloring of a set vectors, support of a coloring, rank partition from [1].

**Theorem 1.1.** [2]. *In  $V_\lambda$  the immanantal decomposable tensor  $x_1 * \dots * x_m$  is nonzero if and only if there exists a  $\lambda'$ -coloring of the family  $x_1, \dots, x_m$ ; if and only if  $\rho(x_1, \dots, x_m) \succeq \lambda'$ , where  $\rho(x_1, \dots, x_m)$  is the rank partition of the family  $x_1, \dots, x_m$ .*

**Theorem 1.2.** [1]. *Let  $(x_1, \dots, x_m)$  and  $(y_1, \dots, y_m)$  be families of nonzero vectors of  $V$ , and let  $\lambda = (\lambda_1, \dots, \lambda_t)$  be an irreducible character of  $S_m$ . Assume that  $x_1 * \dots * x_m \neq 0$ . Then,*

$$x_1 * \dots * x_m = y_1 * \dots * y_m$$

*if and only if the following conditions hold:*

- (1) *The set of supports of the  $\lambda'$ -colorings of  $(x_1, \dots, x_m)$  is equal to the set of supports of the  $\lambda'$ -colorings of  $(y_1, \dots, y_m)$ ; and*
- (2) *If  $\Delta_1, \dots, \Delta_{\lambda_1}$  is a support of a  $\lambda'$ -coloring of  $(x_1, \dots, x_m)$ , then there exists  $\sigma \in S_{\lambda'}$  such that*

$$\langle x_i : i \in \Delta_j \rangle = \langle y_i : i \in \Delta_{\sigma(j)} \rangle, \quad j = 1, \dots, \lambda_1,$$

and

$$\prod_{i=1}^{\lambda_1} \det M[y_1, \dots, y_m | x_1, \dots, x_m][\Delta_{\sigma(i)} | \Delta_i] = 1.$$

In this paper, we generalize previous theorems, for symmetry classes of tensors associated with *Young subgroups*.

## 2. RESULTS

The purpose of this paper is to verify the vanishing problem and equality problem in symmetry classes associated to irreducible characters of the *Young subgroups*.

For a partition  $\mu = (\mu_1, \dots, \mu_r)$  of  $m$ , the corresponding *Young subgroup* of  $S_m$  is

$$S_\mu = S_{M_1} \times \dots \times S_{M_r},$$

where  $S_{M_i}$  is the subgroup of  $S_m$  consisting of those permutations that fix every integer not contained in the set

$$M_i = \{1 \leq k \leq m : \sum_{j=1}^{i-1} \mu_j < k \leq \sum_{j=1}^i \mu_j\}.$$

In this section, we assume that  $G = S_\mu$  and let  $\lambda$  be an irreducible character of  $G$ . Then we have

$$\lambda = \prod_{i=1}^r \lambda^i,$$

where  $\lambda^i$  is an irreducible character of  $S_{M_i} \cong S_{\mu_i}$ .

Let  $x_1, \dots, x_m \in V$  be arbitrary vectors. We set:

$$\begin{aligned} x_{M_1}^\otimes &:= x_1 \otimes \cdots \otimes x_{\mu_1} \\ x_{M_2}^\otimes &:= x_{\mu_1+1} \otimes \cdots \otimes x_{\mu_1+\mu_2} \\ &\vdots \\ x_{M_r}^\otimes &:= x_{\mu_1+\cdots+\mu_{r-1}+1} \otimes \cdots \otimes x_m. \end{aligned}$$

**Lemma 2.1.** *We have*

$$x_1 * \cdots * x_m = T(\lambda^1, S_{M_1})(x_{M_1}^\otimes) \otimes \cdots \otimes T(\lambda^r, S_{M_r})(x_{M_r}^\otimes).$$

**Proof** We know that

$$G = S_{M_1} \times \cdots \times S_{M_r},$$

so the right hand side of the above equation is

$$\frac{\lambda^1(1) \cdots \lambda^r(1)}{\mu_1! \cdots \mu_r!} \sum_{\sigma_1 \in S_{M_1}} \cdots \sum_{\sigma_r \in S_{M_r}} \left( \prod_{i=1}^r \lambda^i \right) (\sigma_1 \cdots \sigma_r) P(\sigma_1)(x_{M_1}^\otimes) \otimes \cdots \otimes P(\sigma_r)(x_{M_r}^\otimes).$$

Let  $\sigma = \sigma_1 \cdots \sigma_r$ . Then the above tensor becomes

$$\frac{\lambda(1)}{|G|} \sum_{\sigma \in G} \lambda(\sigma) P(\sigma)(x_1 \otimes \cdots \otimes x_m),$$

which is just  $x_1 * \cdots * x_m$ .

By using Lemma 2.1 and theorems 1.1, 1.2, we obtain following results.

**Corollary 2.2.** *In  $V_\lambda(G)$  the decomposable tensor  $x_1 * \cdots * x_m$  is nonzero if and only if for every  $1 \leq i \leq r$ , there exists a  $(\lambda^i)'$ -coloring of the family  $(x_j : j \in M_i)$ ; if and only if  $\rho(x_j : j \in M_i) \succeq (\lambda^i)'$ .*

**Corollary 2.3.** *Let  $(x_1, \dots, x_m)$  and  $(y_1, \dots, y_m)$  be families of nonzero vectors of  $V$ . Assume that  $x_1 * \cdots * x_m \neq 0$ . Then,*

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m$$

*if and only if the following conditions hold for all  $1 \leq i \leq r$ :*

- (1) *The set of supports of the  $(\lambda^i)'$ -colorings of  $(x_j : j \in M_i)$  is equal to the set of supports of the  $(\lambda^i)'$ -colorings of  $(y_j : j \in M_i)$ ; and*
- (2) *If  $\Delta_1, \dots, \Delta_{\lambda_1^i}$  is a support of a  $(\lambda^i)'$ -coloring of  $(x_j : j \in M_i)$ , then there*

exists  $\sigma \in S_{(\lambda^i)}$  such that

$$\langle x_j : j \in \Delta_k \rangle = \langle y_j : j \in \Delta_{\sigma(k)} \rangle, \quad k = 1, \dots, \lambda_1^i,$$

and

$$\prod_{k=1}^{\lambda_1^i} \det M[y_j : j \in M_i | x_j : j \in M_i][\Delta_{\sigma(k)} | \Delta_k] = 1.$$

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