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REES MODULES, END OF ASSOCIATED GRADED MODULES AND REDUCTION NUMBER

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ABSTRACT. For a finitely generated module M over a local ring $(A, \mathfrak{m})$, let $R_{\mathfrak{b}}(M) := \bigoplus_{n=0}^{\infty} \mathfrak{b}^n M$ (resp. $G_{\mathfrak{b}}(M) := \bigoplus_{n=0}^{\infty} \mathfrak{b}^n M/\mathfrak{b}^{n+1}M$) denote the Rees module of M associated to the ideal $\mathfrak{b}$ of A (resp. the associated graded module of M with respect to $\mathfrak{b}$). In this talk we study some properties of such modules. Some independence results about the reduction number of $\mathfrak{b}$ relative to M will be investigated. We also prove that for $g = \text{grade}(G(\mathfrak{b})_+, G_{\mathfrak{b}}(M)) < \min\{\text{grade}(\mathfrak{b}, M), \lambda(\mathfrak{b}, M)\}$ the inequality $a_g(G_{\mathfrak{b}}(M)) < a_{g+1}(G_{\mathfrak{b}}(M))$ holds, where $G(\mathfrak{b})_+$ denotes the irrelevant ideal of associated graded ring $G(b)$, $\lambda(\mathfrak{b}, M)$ is the analytic spread of $\mathfrak{b}$ relative to M and $a_i(G_{\mathfrak{b}}(M))$ is the end of graded module $H^i_{G(\mathfrak{b})_+}(G_{\mathfrak{b}}(M))$ for $i = g, g+1$.

1. INTRODUCTION

Let $(A, \mathfrak{m})$ be a local ring. In [2], Marley proved that if A is Cohen-Macaulay, then for an $\mathfrak{m}$-primary ideal b, we have $a_g(G(\mathfrak{b})) < a_{g+1}(G(\mathfrak{b}))$, where $g = \text{grade}G(\mathfrak{b})_+ \leq \dim A - 1$. We will give here an extension of this result to the case of associated graded modules without any assumptions on the ideal $\mathfrak{b}$.
In order to explain the main result of the paper we need some notations.
We denote by $R_{\mathfrak{b}}(M)$ (resp. by $G_{\mathfrak{b}}(M)$) the Rees module of M associated to $\mathfrak{b}$ (resp. the associated graded module of M with respect to $\mathfrak{b}$), namely

$$R_{\mathfrak{b}}(M) := \bigoplus_{n=0}^{\infty} \mathfrak{b}^n M \text{ and } G_{\mathfrak{b}}(M) := \bigoplus_{n=0}^{\infty} \mathfrak{b}^n M/\mathfrak{b}^{n+1}M.$$

In the case where $M = A$ we denote it by $R(\mathfrak{b})$ (resp. by $G(\mathfrak{b})$) and call it the Rees algebra (resp. the associated graded ring) of $\mathfrak{b}$ simply. We denote by $\mathfrak{M}$ the unique homogeneous maximal ideal of $G(\mathfrak{b})$. Then following [1, Definition 4.5.7], the *analytic spread* of $\mathfrak{b}$ relative to M is defined to be $\lambda(\mathfrak{b}, M) = \dim(R_{\mathfrak{b}}(M)/\mathfrak{m}R_{\mathfrak{b}}(M)) = \dim(G_{\mathfrak{b}}(M)/\mathfrak{m}G_{\mathfrak{b}}(M))$. Note that $\text{ht}_M \mathfrak{b} \leq \lambda(\mathfrak{b}, M) \leq d = \dim M$.
For simplicity, we always assume that the residue field $k = A/\mathfrak{m}$ of A is infinite.

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Recall from [1, Definition 4.6.4] that an ideal $\mathfrak{a} \subseteq \mathfrak{b}$ is called a *reduction* of $\mathfrak{b}$ relative to M if $\mathfrak{a}\mathfrak{b}^{n-1}M = \mathfrak{b}^n M$ for some positive integer n. We denote by $r_{\mathfrak{a}}(\mathfrak{b}, M)$ the least integer n with this property. A reduction $\mathfrak{a}$ of $\mathfrak{b}$ relative to M is called a *minimal reduction* if it does not properly contain any other reduction of $\mathfrak{b}$ relative to M. Since k is infinite, it is well known that the minimal reductions relative to M always exist (see [4, Section 4]). We define the *reduction number* of $\mathfrak{b}$ relative to M by

$$r(\mathfrak{b}, M) = \min\{r_{\mathfrak{a}}(\mathfrak{b}, M) : \mathfrak{a} \text{ is a minimal reduction of } \mathfrak{b} \text{ relative to } M\}.$$

2. Main results

Theorem 2.1. *Let $ht_M \mathfrak{b} > 0$. Then $R(M)$ is a Cohen-Macaulay R-module if and only if $[H^i_{\mathfrak{M}}(G_{\mathfrak{b}}(M))]_n = 0$ for all $i < d, n \neq -1$ and end$(H^d_{\mathfrak{M}}(G_{\mathfrak{b}}(M))) <$ 0.*

Corollary 2.2. *Let $\lambda(\mathfrak{b}, M) = ht_M \mathfrak{b} = 0$ or d and assume that $R_{\mathfrak{b}}(M)$ is Cohen-Macaulay. Then $r(\mathfrak{b}, M) = reg(G_{\mathfrak{b}}(M)$, that is the reduction number of $\mathfrak{b}$ relative to M is independent of the choice of minimal reduction.*

Proof. The case $\lambda(\mathfrak{b}, M) = 0$ is clear. So let $\lambda(\mathfrak{b}, M) = d$. Let $\mathfrak{a} = (a_1, ..., a_\lambda) \subseteq \mathfrak{b}$ be a minimal reduction of $\mathfrak{b}$ relative to M. Then $\mathfrak{a}^* = (a_1^*, ..., a_\lambda^*)$ is a minimal reduction of $G(\mathfrak{b})_+$ relative to $G_{\mathfrak{b}}(M)$ and $r_{\mathfrak{a}}(\mathfrak{b}, M) = r_{\mathfrak{a}^*}(G(\mathfrak{b})_+, G_{\mathfrak{b}}(M))$. We note that by Theorem 2.1 $[H^i_{\mathfrak{M}}(G_{\mathfrak{b}}(M))]_n = 0$ for all $i < d$ and $n \neq -1$. Then by [4, Corollary 3.12], the ideal $G(\mathfrak{b})_+$ is standard relative to $G_{\mathfrak{b}}(M)$. Therefore in view of [4, Proposition 3.1], $a_1^*, ..., a_\lambda^*$ is a d-sequence of $G_{\mathfrak{b}}(M)$. Hence we get $r_{\mathfrak{a}^*}(G(\mathfrak{b})_+, G_{\mathfrak{b}}(M)) = \text{reg}(G_{\mathfrak{b}}(M))$. So $r_{\mathfrak{a}}(\mathfrak{b}, M) = \text{reg}(G_{\mathfrak{b}}(M))$ and the result follows in this case.

Lemma 2.3. *Let $\mathfrak{a}$ be a minimal reduction of $\mathfrak{b}$ relative to M and let $x \in \mathfrak{a} \backslash \mathfrak{m}\mathfrak{a}$ such that x^* is a $G(M)$-regular element. Then $\lambda(\mathfrak{b}/(x), M/xM) = \lambda(\mathfrak{b}, M) - 1$.*

Theorem 2.4. *Let $g =$ grade $(G(\mathfrak{b})_+, G_{\mathfrak{b}}(M)) <$ min$\{$grade$(\mathfrak{b}, M), \lambda(\mathfrak{b}, M)\}$. Then $a_g(G(M)) < a_{g+1}(G(M))$.*

Proof. We proceed by induction on $g \geq 0$. In the case $g = 0$, we consider the exact sequences

$$0 \longrightarrow R_{\mathfrak{b}}(M)_+ \longrightarrow R_{\mathfrak{b}}(M) \longrightarrow M \longrightarrow 0,$$

of graded $R(\mathfrak{b})$-modules. Since $[H^i_{R(\mathfrak{b})_+}(M)]_n = 0$ for $n \neq 0$,

$$[H^i_{R(\mathfrak{b})_+}(R_{\mathfrak{b}}(M)_+)(1)]_{n-1} = [H^i_{R(\mathfrak{b})_+}(R_{\mathfrak{b}}(M)_+)]_n \cong [H^i_{R(\mathfrak{b})_+}(R_{\mathfrak{b}}(M))]_n$$

for $n \neq 0$. Now from the exact sequence

$$0 \longrightarrow R_{\mathfrak{b}}(M)_+(1) \longrightarrow R_{\mathfrak{b}}(M) \longrightarrow G_{\mathfrak{b}}(M) \longrightarrow 0,$$

and using the fact that $H^0_{R(\mathfrak{b})_+}(R_{\mathfrak{b}}(M)) = 0$ (because of grade$(\mathfrak{b}, M) > 0$), we get the exact sequence

$$0 \to [H^0_{R(\mathfrak{b})_+}(G_{\mathfrak{b}}(M))]_n \to [H^1_{R(\mathfrak{b})_+}(R_{\mathfrak{b}}(M))]_{n+1} \to$$

$$(1) \qquad [H^1_{R(\mathfrak{b})_+}(R_\mathfrak{b}(M))]_n \to [H^1_{R(\mathfrak{b})_+}(G_\mathfrak{b}(M))]_n.$$

Set $a = a_1(G_\mathfrak{b}(M)) = \operatorname{end}(H^1_{R(\mathfrak{b})_+}(G_\mathfrak{b}(M)))$. (note that $H^i_{G_+}(G_\mathfrak{b}(M)) \cong H^i_{R(\mathfrak{b})_+}(G_\mathfrak{b}(M))$ for all $i \geq 0$.) Then from the exact sequence (1) we have the epimorphism

$$[H^1_{R(\mathfrak{b})_+}(R_\mathfrak{b}(M))]_{n+1} \longrightarrow [H^1_{R(\mathfrak{b})_+}(R_\mathfrak{b}(M))]_n \longrightarrow 0,$$

for all $n > a$. But we have $[H^i_{R(\mathfrak{b})_+}(R_\mathfrak{b}(M))]_n = 0$ for large values of n, hence $[H^1_{R(\mathfrak{b})_+}(R_\mathfrak{b}(M))]_n = 0$ for all $n > a$. We claim that $a > 0$. Suppose the contrary $a \leq 0$. Then it follows from the exact sequence (1) that $[H^0_{R(\mathfrak{b})_+}(G_\mathfrak{b}(M))]_n = 0$ for all $n \geq 0$. This yields the contradiction $H^0_{G(\mathfrak{b})_+}(G_\mathfrak{b}(M)) = H^0_{R(\mathfrak{b})_+}(G_\mathfrak{b}(M)) = 0$. (Here it should be noted that $H^0_{R(\mathfrak{b})_+}(G_\mathfrak{b}(M))$ as a submodule of $G_\mathfrak{b}(M)$ has no negative homogeneous components and because of $g = 0$ we have

$$H^0_{G(\mathfrak{b})_+}(G_\mathfrak{b}(M)) \neq 0.)$$

So $a > 0$. If we consider once again the sequence (1) we see that

$$[H^0_{R(\mathfrak{b})_+}(G_\mathfrak{b}(M))]_n = 0$$

for $n \geq a$, which gives that $a_0(G_\mathfrak{b}(M)) \leq a - 1 < a_1(G_\mathfrak{b}(M))$. So the result follows in the case $g = 0$.

Now suppose that $g > 0$ and that the result has been proved for smaller values of g. Let $\mathfrak{a}$ be a minimal reduction of $\mathfrak{b}$ relative to M and let $x \in \mathfrak{a} \setminus \mathfrak{m}\mathfrak{a}$ such that $x^* \in G(\mathfrak{b})_+$ is a $G_\mathfrak{b}(M)$-regular element. Then by the previous lemma we have $\lambda(\mathfrak{b}/(x), M/xM) = \lambda(\mathfrak{b}, M) - 1$. Furthermore

$$\operatorname{grade}(G(\mathfrak{b}/(x))_+, G_{\mathfrak{b}/(x)}(M/xM)) = g - 1$$

and $\operatorname{grade}(\mathfrak{b}/(x), M/xM) \geq g$. We consider the exact sequence

$$0 \to G_\mathfrak{b}(M)(-1) \xrightarrow{x^*} G_\mathfrak{b}(M) \to G_{\mathfrak{b}/(x)}(M/xM) \cong G_\mathfrak{b}(M)/x^*G_\mathfrak{b}(M) \to 0,$$

to deduce the exact sequence

$$0 \to [H^{g-1}_{G(\mathfrak{b})_+}(G_\mathfrak{b}(M)/x^*G_\mathfrak{b}(M))]_n \to [H^g_{G(\mathfrak{b})_+}(G_\mathfrak{b}(M))]_{n-1} \to [H^g_{G(\mathfrak{b})_+}(G_\mathfrak{b}(M))]_n$$
$$\to [H^g_{G(\mathfrak{b})_+}(G_\mathfrak{b}(M)/x^*G_\mathfrak{b}(M))]_n \to [H^{g+1}_{G(\mathfrak{b})_+}(G_\mathfrak{b}(M))]_{n-1},$$

of $A/\mathfrak{b}$-modules.

Now by the induction hypothesis, $a_{g-1}(G_\mathfrak{b}(M)/x^*G_\mathfrak{b}(M)) < a_g(G_\mathfrak{b}(M)/x^*G_\mathfrak{b}(M))$. Let $n \geq a_g(G_\mathfrak{b}(M)/x^*G_\mathfrak{b}(M))$. Then from the above exact sequence we get the monomorphism

$$0 \longrightarrow [H^g_{G(\mathfrak{b})_+}(G_\mathfrak{b}(M))]_{n-1} \to [H^g_{G(\mathfrak{b})_+}(G_\mathfrak{b}(M))]_n,$$

which gives that $H^g_{G(\mathfrak{b})_+}(G_\mathfrak{b}(M))]_{n-1} = 0$ for all $n \geq a_g(G_\mathfrak{b}(M)/x^*G_\mathfrak{b}(M))$. This means that $a_g(G_\mathfrak{b}(M)) < a_g(G_\mathfrak{b}(M)/x^*G_\mathfrak{b}(M)) - 1$. Therefore, it follows from the above exact sequence that $[H^{g+1}_{G(\mathfrak{b})_+}(G_\mathfrak{b}(M))]_{a'-1} \neq 0$, where $a' = a_g(G_\mathfrak{b}(M)/x^*G_\mathfrak{b}(M))$. So $a_g(G_\mathfrak{b}(M)) < a_g(G_\mathfrak{b}(M)/x^*G_\mathfrak{b}(M)) - 1 \leq a_{g+1}(G_\mathfrak{b}(M))$ and the desired result follows.

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