

ON GENERALIZED LIFTING MODULES

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ABSTRACT. Generalized relative injectivity was renamed as *ojective* by Mohamed and Muller [4]. They also studied the concept of ojective modules in two ways, namely to cojective and *cojective modules [5]. Here we introduce the concept of M -pjectivity, which is a generalization of M -*cojectivity, and give some relations between pjective and lifting modules. We also study the concept of generalized lifting modules, and give some properties of such modules.

1. INTRODUCTION

Throughout this talk R will denote an arbitrary associative ring with identity and M will be a unital right R -module. The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M ; $N \ll M$ means that N is small in M (i.e. $\forall L \lesssim M, L+N \neq M$). A module M is called *hollow* if every proper submodule of M is small in M . A module N is said to be *small* if $N \ll L$ for some module L . For $N, L \leq M$, N is *supplement* of L in M if $N+L = M$ with $N \cap L \ll N$. A module M is called *supplemented* if every submodule of M has a supplement in M . On the other hand, the module M is *amply supplemented* if, for any submodules A, B of M with $M = A + B$ there exists a supplement P of A in M such that $P \leq B$. The module M is called *\oplus -supplemented* if every submodule of M has a supplement that is a direct summand of M .

Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then B is called a *cosmall* submodule of A in M . A submodule A of M is called *coclosed* if A has no proper cosmall submodule.

A module M is *lifting* if for every submodule A of M , there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A \ll M$. Recall that module M is a (D_3) -module, if $M = A + B$, where A and B are direct summands of M , then $A \cap B$ is a direct summand of M . We say that a module M has *hollow dimension* n , if there exists a small epimorphism from M to a direct sum of n hollow modules.

Let M_1, M_2 be modules. Recall that the module M_1 is *small M_2 -projective* if every homomorphism $f : M_1 \rightarrow M_2/A$, where A is a submodule of M_2 and $Im f \ll M_2/A$, can be lifted to a homomorphism $\phi : M_1 \rightarrow M_2$.

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A module N is M -*pjjective* if every supplement of M in $M \oplus N$ is a direct summand. If N is M -*pjjective* and M is N -*pjjective*, we say that N and M are relatively *pjjective*. The problem of finding a satisfactory necessary and sufficient condition for a direct sum of lifting modules to be lifting is still open. We show that relative *pjjectivity* is necessary and sufficient for a direct sum of two lifting modules to be lifting. We also introduce the concept of generalized lifting modules, and give some properties of such modules in analogy with properties for lifting modules.

2. M - P JECTIVE MODULES

Definition 2.1. Let $M = A \oplus B$. Then B is called A -*pjjective* if every supplement C of A in M is a direct summand.

Lemma 2.2. Let $M = M_1 \oplus M_2$ and $N, L \leq M_1$. If N is a supplement of L in M_1 , then

- (1) $N \oplus M_2$ is a supplement of L in M .
- (2) N is a supplement of $L \oplus M_2$ in M .

Lemma 2.3. Let $M = A \oplus B$, where B is A -*pjjective*. If $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$. Then (for $i, j = 1, 2$);

- (1) B_i is A -*pjjective*;
- (2) B is A_j -*pjjective*;
- (3) B_i is A_j -*pjjective*.

Definition 2.4. A module M is called an *absolute relative pjjective* module (for short *ARPJ*-module) if M_i is M_j -*pjjective* ($i \neq j$); whenever $M = M_1 \oplus M_2$.

Clearly every lifting module is an *ARPJ*-module and any indecomposable module is obviously an *ARPJ*-module, which is not lifting.

Theorem 2.5. The following are equivalent for a module M :

- (1) M is a lifting module;
- (2) M is an amply supplemented, \oplus -supplemented and *ARPJ*-module.

Proposition 2.6. Let $M = M_1 \oplus M_2$. If M_1 is M_2 -*projective*, then M_1 is M_2 -*pjjective*.

Theorem 2.7. Let $M = M_1 \oplus M_2$ be an amply supplemented (D_3)-module. If M_1 is M_2 -*projective*, then M_1 is M_2 -*projective*.

Lemma 2.8. Let $M = A \oplus B$ where A is B -*pjjective* and B is lifting. If X is a coclosed submodule of M with $M = X + B$, then X is a summand of M .

Theorem 2.9. Let $M = M_1 \oplus M_2$ be an amply supplemented module. Then M is lifting if and only if the M_i is lifting, and is M_j -*pjjective*, $i \neq j (= 1, 2)$.

3. GL -MODULES

Definition 3.1. A module M is called a *generalized* lifting module (for short a *GL*-module) if the following condition satisfied: If $M = M_1 \oplus M_2$ and $A \leq M$, then there exist $C_i \leq^\oplus M_i$ ($i = 1, 2$) such that $C_1 \oplus C_2$ is a supplement of A in M .

Theorem 3.2. (cf. [3]) *For any ring R , any finite direct sum of \oplus -supplemented R -modules is \oplus -supplemented.*

Remark 3.3. In the proof of the above Theorem, we obtain that for any submodule L of $M = M_1 \oplus M_2$, there exists a supplement $K \oplus H$ of L , where $K \leq^\oplus M_1$ and $H \leq^\oplus M_2$. Therefore we have the following Corollary.

Corollary 3.4. *The following are equivalent for a module M .*

- (1) M is a GL -module.
- (2) Every direct summand of M is a \oplus -supplemented module.

Corollary 3.5. *Let $M = \bigoplus_{i=1}^n M_i$ be a finite direct sum of modules. If each M_i is \oplus -supplemented, then M is GL -module.*

Corollary 3.6. *Direct summands of a GL -module are GL -modules.*

Corollary 3.7. *Every lifting module is a GL -module.*

Example 3.8. Let p be any prime integer. \mathbb{Z} -Module $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not lifting, but is GL -module.

Theorem 3.9. *Let M be \oplus -supplemented and finite hollow dimensional module. Then M is a GL -module.*

The following implications are now clear for a module M :

Lifting module \Rightarrow GL -module \Rightarrow \oplus -Supplemented module.

Proposition 3.10. *The following statements are equivalent for a module $M = \bigoplus_{i=1}^n M_i$.*

- (1) The $M_i (i = 1, 2, \dots, n)$ is \oplus -supplemented.
- (2) Each submodule of M has a supplement in M of the form $\bigoplus_{i=1}^n N_i$, where $N_i \leq^\oplus M_i (i = 1, 2, \dots, n)$.

Proposition 3.11. *If M is a GL -module with finite hollow dimension, then M is a finite direct sum of hollow submodules.*

Remark 3.12. Consider a direct sum of hollow modules, which contains an indecomposable and not hollow summand submodule. This module is \oplus -supplemented, which is not a GL -module. This also shows that direct summands of \oplus -supplemented modules need not be \oplus -supplemented.

Theorem 3.13. *If M is a \oplus -supplemented, and satisfies in this condition that for every two direct summands N_1 and N_2 of M such that $N_1 \cap N_2$ is coclosed in M , implies that $N_1 \cap N_2$ is a direct summand of M . Then M is a GL -module.*

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