On a construction of Local Cohomology Functors for Triangulated Categories [1]

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Abstract

Let $\mathcal{T}$ be a compactly generated triangulated category which is closed under arbitrary small coproducts. Based on a construction of local cohomology functors on triangulated categories, which is introduced recently by Benson, Iyengar and Krause, a notion of support is assigned to any object X of $\mathcal{T}$. Using this new notion of support, we will study local cohomology functors and show that in special cases, our results recover and generalize the known results about the usual local cohomology functors.

1 Introduction

Let $\mathcal{T}$ be a triangulated category. We say that $\mathcal{T}$ satisfies [TR5] if it has arbitrary small coproducts. An object C of $\mathcal{T}$ is called compact if the functor $\mathrm{Hom}_{\mathcal{T}}(C, \ )$ preserves small coproducts. A set $\mathcal{G}$ of objects of $\mathcal{T}$ is called a generating set for $\mathcal{T}$ if for each $X \in \mathcal{T}$, there exists an object G in $\mathcal{G}$ such that $\mathrm{Hom}_{\mathcal{T}}(G, X) \neq 0$. $\mathcal{T}$ is called compactly generated if it has a generating set of compact objects.

Assume that $\mathcal{T}$ is a [TR5] compactly generated triangulated category with the graded center $\mathcal{Z}(\mathcal{T})$. Let R be a graded-commutative noetherian ring and $\phi : R \longrightarrow \mathcal{Z}(\mathcal{T})$ be a homomorphism of graded rings. This, in particular, implies that for any objects X and Y of $\mathcal{T}$, the abelian group $\mathrm{Hom}^*_{\mathcal{T}}(X, Y)$ has a structure of a graded R-module. If $X = C$ is a compact object, then the R-module $\mathrm{Hom}^*_{\mathcal{T}}(C, Y)$ will be denoted by $\mathrm{H}^*_C(Y)$ and is called the cohomology of Y with respect to C.

In [1] a notion of (small) support is assigned to any object X of $\mathcal{T}$. Assume that for an R-module M, $\mathrm{supp}_R M$ denotes the small (or cohomological) support of M. By [1, Theorem 5.2] the support of X, denoted $\mathrm{supp}_R X$, is equal to the

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set $\bigcup_{C\in\mathcal{T}^c} \min \operatorname{supp}_R \mathrm{H}^*_C(X)$, where $\mathcal{T}^{\mathrm{c}}$ denotes the full subcategory of $\mathcal{T}$ formed by all compact objects.

In this talk, we introduce a (big) support for X and show that the construction of local cohomology functors can be studied well using this new notion of support. We will study some properties of local cohomology functors and get some results that in special cases, recover and generalize the known results about the usual local cohomology functors.

2 Preliminaries

Throughout R denotes a graded-commutative noetherian ring and $\operatorname{Spec}(R)$ denotes the set of graded prime ideals of R. For a point $\mathfrak{p}$ in $\operatorname{Spec}(R)$, $R_{\mathfrak{p}}$ denotes the homogeneous localization of R, which is a graded local ring.

Let $\mathcal{T}$ be a compactly generated triangulated category. For any objects X and Y of $\mathcal{T}$, let $\operatorname{Hom}_{\mathcal{T}}(X,Y)$ denote the abelian group of morphisms and $\operatorname{Hom}^*_{\mathcal{T}}(X,Y)$ denote the graded abelian group $\bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(X,\Sigma^i Y)$. It is clearly a right $\operatorname{End}^*_{\mathcal{T}}(X)$-module and a left $\operatorname{End}^*_{\mathcal{T}}(Y)$-module, where for any object X of $\mathcal{T}$, $\operatorname{End}^*_{\mathcal{T}}(X) = \operatorname{Hom}^*_{\mathcal{T}}(X,X)$.

Let $Z(\mathcal{T})$ denote the graded center of $\mathcal{T}$ with

$$Z(\mathcal{T})^n = \{\eta : \mathrm{Id}_{\mathcal{T}} \to \Sigma^n \mid \eta\Sigma = (-1)^n \Sigma\eta\},$$

for any integer n. $Z(\mathcal{T})$ is a graded commutative ring.

Now we fix a graded commutative noetherian ring R and a homomorphism of graded rings $\phi : R \to Z(\mathcal{T})$. This implies that any graded abelian group $\operatorname{Hom}^*_{\mathcal{T}}(X,Y)$ is an R-module with the action induced by ϕ; for details see [1, Section 4]. Therefore $\mathcal{T}$ becomes an R-linear triangulated category.

Let $\mathcal{U}$ be a subset of $\operatorname{Spec}(R)$. The specialization closure of $\mathcal{U}$, denoted $\mathrm{cl}\mathcal{U}$, is defined by

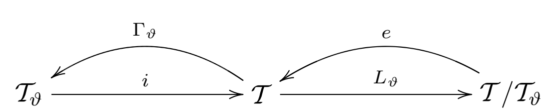
$$\mathrm{cl}\mathcal{U} = \{\mathfrak{p} \in \mathrm{Spec(R)} \mid \mathfrak{p} \supseteq \mathfrak{q},\ \text{for some } \mathfrak{q} \in \mathcal{U}\}.$$

A subset $\mathcal{U}$ of $\operatorname{Spec}(R)$ is called specialization closed if $\mathrm{cl}\mathcal{U} = \mathcal{U}$.

Let ϑ be a specialization closed subset of $\operatorname{Spec}(R)$. Set

$$\mathcal{T}_\vartheta = \{X \in \mathcal{T} \mid \operatorname{supp}_R \mathrm{H}^*_C(X) \subseteq \vartheta \text{ for any } \mathrm{C} \in \mathcal{T}^{\mathrm{c}}\}.$$

By [1, Lemma 4.3], $\mathcal{T}_\vartheta$ is a localizing subcategory of $\mathcal{T}$, i.e. it is closed under direct summands and small coproducts. So we have the localization functor $L_\vartheta : \mathcal{T} \to \mathcal{T}$. It induces an equivalence of categories $\mathcal{T}/\mathrm{Ker}L_\vartheta \cong \mathrm{Im}L_\vartheta$, where $\mathcal{T}/\mathrm{Ker}L_\vartheta$ denotes the Verdier quotient of $\mathcal{T}$ with respect to $\mathrm{Ker}L_\vartheta$ and $\mathrm{Im}L_\vartheta$ is the essential image of L_ϑ. Moreover, $L_\vartheta X = 0$ if and only if $X \in \mathcal{T}_\vartheta$. Hence we get a localization sequence of triangulated functors



in which Γ_ϑ is the right adjoint of the inclusion functor i and the inclusion functor e is the right adjoint of the localization functor L_ϑ on $\mathcal{T}$. One can consult [2, §II.2] for studying the properties of a localization sequence of functors. In particular, for any X in $\mathcal{T}$, we have an exact triangle

$$\Gamma_\vartheta X \longrightarrow X \longrightarrow L_\vartheta X \rightsquigarrow ,$$

in which Γ_ϑ is the right adjoint of the inclusion functor $\mathcal{T}_\vartheta \hookrightarrow \mathcal{T}$. $\Gamma_\vartheta X$ is then called the local cohomology of X supported on ϑ, see [1, Section 4].

3 Main results

Let $\mathcal{T}$ be a compactly generated triangulated category which is R-linear, where as before R is a graded-commutative noetherian ring.

Let $\mathfrak{p}$ be a prime ideal of R. Since $\mathcal{Z}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec}(R) \mid \mathfrak{q} \nsubseteq \mathfrak{p}\}$ is a specialization closed subset of $\operatorname{Spec}(R)$, there is a localization functor $L_{\mathcal{Z}(\mathfrak{p})} : \mathcal{T} \to \mathcal{T}$. For any $X \in \mathcal{T}$, let $X_\mathfrak{p}$ denote the object $L_{\mathcal{Z}(\mathfrak{p})}X$ and $\mathcal{T}_\mathfrak{p}$ denote the essential image of $L_{\mathcal{Z}(\mathfrak{p})}$. So $\mathcal{T}_\mathfrak{p}$ is the full subcategory of $\mathcal{T}$ formed by all objects isomorphic to an object of the form $L_{\mathcal{Z}(\mathfrak{p})}X$, for some $X \in \mathcal{T}$. Note that $\mathcal{T}_\mathfrak{p}$ is the Verdier quotient $\mathcal{T}/\mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$. Moreover, by [1, Theorem 6.4], $\mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$ is compactly generated in $\mathcal{T}$ and so by Neeman-Ravenel-Thomason localization theorem $\mathcal{T}_\mathfrak{p}$ is compactly generated.

Definition 3.1. *Let X be an object of $\mathcal{T}$. We define the (big) support of X, denoted* $\operatorname{Supp}_R X$*, to be the set*

$$\operatorname{Supp}_R X = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid X_\mathfrak{p} \neq 0\}.$$

This definition of support for an object is completely related to the usual support of the cohomology of objects. Following theorem establishes this fact.

Theorem 3.2. *Let X be an object of $\mathcal{T}$. Then*

$$\operatorname{Supp}_R X = \bigcup_{C \in \mathcal{T}^c} \operatorname{Supp}_R \mathrm{H}^*_C(X).$$

In particular, for any object X of $\mathcal{T}$ we have $X = 0$ if and only if $\operatorname{Supp}_R X = \emptyset$.

Theorem 3.3. *Let C be a compact object of $\mathcal{T}$. Then for any object X of C and any prime ideal $\mathfrak{p}$ of R, there exists an isomorphism*

$$\mathrm{H}^*_C(X)_\mathfrak{p} \cong \mathrm{H}^*_{C_\mathfrak{p}}(X_\mathfrak{p})$$

of $R_\mathfrak{p}$-modules.

Now, let X be an object of $\mathcal{T}$ and C be a compact object. Since $\mathrm{H}^*_C(X)$ is a graded R-module, we may consider two invariants

$$\begin{aligned} \inf\nolimits_C X &= \inf(\mathrm{H}^*_C(X)) = \inf\{n \in \mathbb{Z} \mid \mathrm{H}^n_C(X) \neq 0\} \\ \sup\nolimits_C X &= \sup(\mathrm{H}^*_C(X)) = \sup\{n \in \mathbb{Z} \mid \mathrm{H}^n_C(X) \neq 0\}. \end{aligned}$$

We say that an object X of $\mathcal{T}$ is cohomologically bounded above (respectively, bounded below) if, for any compact object C, there exists a positive integer $n(C)$ such that $\sup_C X \leq n(C)$ (respectively, $\inf_C X \geq -n(C)$). X is called cohomologically bounded if it is both cohomologically bounded above and cohomologically bounded below. Let $\mathcal{T}^-$ (respectively, $\mathcal{T}^+, \mathcal{T}^b$) denotes the full subcategory of $\mathcal{T}$, consisting of all cohomologically bounded above (respectively, bounded below, bounded) objects.

Definition 3.4. *Let X be an object of $\mathcal{T}$. We define the dimension of X to be*

$$\dim_R X = \sup\{\dim_R \mathrm{H}^*_C(X) \mid C \in \mathcal{T}^c\}.$$

Obviously, for any object X of $\mathcal{T}$, we have $\dim_R X = \dim_R \Sigma X$.

Definition 3.5. *(1) Let $X \neq 0$ be an object of $\mathcal{T}^-$ and $\mathfrak{a}$ be a graded ideal of R. We define the cohomological dimension of X with respect to $\mathfrak{a}$, denoted* $\mathrm{cd}(\mathfrak{a}, X)$*, to be*

$$\mathrm{cd}(\mathfrak{a}, X) = \sup\{\sup_C \Gamma_{\mathcal{V}(\mathfrak{a})} X - \sup_C X \mid C \text{ is a compact object of } \mathcal{T}\}.$$

(2) Let $X \neq 0$ be an object of $\mathcal{T}^+$ and $\mathfrak{a}$ be a graded ideal of R. We define the cohomological grade of X with respect to $\mathfrak{a}$, denoted $\mathrm{cg}(\mathfrak{a}, X)$*, to be*

$$\mathrm{cg}(\mathfrak{a}, X) = \inf\{\inf_C \Gamma_{\mathcal{V}(\mathfrak{a})} X - \inf_C X \mid C \text{ is a compact object of } \mathcal{T}\}.$$

Note that, for any non-zero object X of $\mathcal{T}^-$ (respectively, $\mathcal{T}^+$), $\mathrm{cd}(\mathfrak{a}, \Sigma X) = \mathrm{cd}(\mathfrak{a}, X)$ (respectively, $\mathrm{cg}(\mathfrak{a}, \Sigma X) = \mathrm{cg}(\mathfrak{a}, X)$).

Theorem 3.6. *Let $\mathfrak{a}$ be a graded ideal of R and $X \neq 0$ be an object of $\mathcal{T}^-$. Then* $\mathrm{cd}(\mathfrak{a}, X) \leq \mathrm{ara}(\mathfrak{a})$.

Theorem 3.7. *Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{a} \subseteq \mathfrak{m}$ be an ideal of R. Then for any cohomologically finite object $0 \neq X \in \mathcal{T}^+$, we do have*

$$\mathrm{grade}(\mathfrak{a}, X) \leq \mathrm{cg}(\mathfrak{a}, X),$$

where $\mathrm{grade}(\mathfrak{a}, X)$ *is defined to be* $\mathrm{grade}(\mathfrak{a}, \mathrm{H}^*_C(X))$.

Theorem 3.8. *Let $\mathfrak{a}$ be an ideal of R and $X \in \mathcal{T}$ be cohomologically finite. Then for any integer i we have*

$$\dim_R \mathrm{H}^i_C(\Gamma_{\mathcal{V}(\mathfrak{a})} X) + i \leq \dim_R X + \sup_C X.$$

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