Cleanness and shellablity[1]

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Abstract

Let R be a Noetherian ring and M an R-module. A chain

$$\mathcal{F}:\ (0) = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

of submodules of M is called a prime filtration of M, if for all $i = 1, \ldots, r$, there exists a prime ideal $P_i \in \mathrm{Spec}\,(R)$ such that $M_i/M_{i-1} \cong R/P_i$. If M is finitely generated such a prime filtration of M always exists, see [4, Theorem 6.4]. The set of prime ideals $P_1, \ldots, P_r$ which define the cyclic quotients of $\mathcal{F}$ will be denoted by $\mathrm{Supp}\,(\mathcal{F})$. It follows from [4, Theorem 6.3] that if $\mathcal{F}$ is a prime filtration of M, then

$$\mathrm{Ass}\,(M) \subset \mathrm{Supp}\,(\mathcal{F}) \subset \mathrm{Supp}\,(M).$$

Let $\mathrm{Min}\,(M)$ denote the set of minimal prime ideals in $\mathrm{Supp}\,(M)$. Dress [2] called the prime filtration $\mathcal{F}$ *clean* if $\mathrm{Supp}\,(\mathcal{F}) = \mathrm{Min}\,(M)$. The R-module M is called clean if it has a clean filtration.

Herzog and Popescu [3] generalized this concept and they called a prime filtration $\mathcal{F}$ *pretty clean*, if for all $i < j$ which $P_i \subseteq P_j$ it follows that $P_i = P_j$. The R-module M is called pretty clean if it admits a pretty clean filtration. It follows from [3, Corollary 3.4] that if $\mathcal{F}$ is a pretty clean filtration of M, then $\mathrm{Supp}\,(\mathcal{F}) = \mathrm{Ass}\,(M)$. The converse of the above fact is not true, see [3, Example 4.4]. We call an R-module M *almost clean* if it admits a prime filtration $\mathcal{F}$ with

$$\mathrm{Supp}\,(\mathcal{F}) = \mathrm{Ass}\,(M).$$

It is easy to see that

$$\text{clean} \Rightarrow \text{pretty clean} \Rightarrow \text{almost clean},$$

and if an R-module M has no embedded associated prime ideals, then

$$M \text{ is clean} \Leftrightarrow M \text{ is pretty clean} \Leftrightarrow M \text{ is almost clean}.$$

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It is well known that for an R-module M one has

$$\operatorname{depth}(M) \leq \min\{\dim(R/P) : P \in \operatorname{Ass}(M)\}.$$

Prime filtration of a module has a nice property. We could prove the following proposition

Proposition 0.1 *Let $\mathcal{F}$ a prime filtration of M such that R/P is Cohen–Macaulay ring for all $P \in Supp(\mathcal{F})$. Then*

$$depth(M) \geq \min\{dim(R/P) : P \in Supp\,\mathcal{F}\}.$$

If we combine the above Proposition with the well known fact which we mentioned above we get

Theorem 0.2 *Let $\mathcal{F}$ be an almost clean filtration of the R-module M such that R/P is Cohen–Macaulay for $P \in Supp(\mathcal{F})$. Then*

$$depth(M) = \min\{dim(R/P) : P \in Ass(M)\}.$$

Moreover, if $dim(R/P) = dim(M)$ for all $P \in Supp(\mathcal{F})$, then M is clean and Cohen–Macaulay.

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables. Let I be a monomial ideal in S. We say that I is pretty clean if S/I is pretty clean. By [5, Corollary 3.11] I is pretty clean if and only if there exist a prime filtration $\mathcal{F}$ of I with the property that height P_i is greeter or equal to height P_{i+1}. From these results we can show that if S/I is pretty clean, then S/I is sequentially Cohen–Macaulay.

Let $I^p \subset T$ be the polarization of I, where T is a polynomial ring in more variables. It was shown in [5] that S/I is pretty clean if and only if T/I^p is clean. Cleanness is the algebraic counterpart of shellability for simplicial complexes. Indeed, let Δ be a simplicial complex and K a field. Dress [2] showed that Δ is (non-pure) shellable in the sense of Bjrner and Wachs [1], if and only if the Stanley–Reisner ring $K[\Delta]$ is clean. We give a very simple proof of Dress's theorem. Also as a corollary of our result about cleanness we dive an easy proof for the well known fact that if Δ is a pure shellable simplicial complex, then $K[\Delta]$ is Cohen–Macaulay.

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