*Tarbiat Moallem University, $20^{th}$ Seminar on Algebra,*
*2-3 Ordibehesht, 1388 (Apr. 22-23, 2009) pp 102-105*

# CONSTRUCTION OF LOCALLY EXTENDED AFFINE LIE ALGEBRAS

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Abstract. In this paper we construct all locally extended affine Lie algebras in general. We describe a construction, which starts with any centerless $(Q(\Delta)\oplus\Lambda)-$graded Lie algebra $\mathcal{L}$ with $\Delta$ a locally irreducible finite root system and $\Lambda$ a torsion- free abelian group, and produce a locally extended affine Lie algebra $L$ with centerless core $\mathcal{L}$.

## 1. Introduction

In 2006, Morita and Yoshii [3], introduced a general version of extended affine Lie algebras, called locally extended affine Lie algebras (LEALAs for short). The structure of a LEALA is understood by coordinatizing its core. The core of a LEALA $L$ is the ideal $L_c$ generated by all subspace $L_\alpha,\ \alpha\in R^\times$, where $R$ is the root system of $L$. We then have a Lie algebra homomorphism $L\longrightarrow Der_{\mathbb{F}}(L_c)$, given by $x\longmapsto adx|_{L_c}$, and we say that $L$ is tame if the kernel of this map lies in $L_c$. So, as in [4], we can recover $L$ from $L_c$ and its derivations.

Throughout this paper we will assume that $\mathbb{F}$ is a field of characteristic zero. We begin by recalling the definition of a locally extended affine Lie algebra, which is the one given in [3]. Throughout, $(L,H)$ will be a toral pair over $\mathbb{F}$, i.e, a non-zero Lie algebra $L$ with a toral subalgebra $H$. We will denote the set of its roots by $R$. Then $L$ has a root space decomposition with respect to $H$ by $L_\alpha=\{x\in L\ :\ [x,h]=\alpha(x)x \text{ for all } h\in H\}$ and $R=\{\alpha\in H^*\ :\ L_\alpha\neq 0\}$.

**Definition 1.1.** A toral pair $(L,H)$ is called a *locally extended affine Lie algebra* (for short LEALA), if it satisfies the following axioms

(LEA1) *$H$ is self-centralizing.*
(LEA2) *$L$ has an invariant nondegenerate symmetric bilinear form $\digamma$.*
(LEA3) *$R\subseteq H^*_\digamma$, where $H^*_\digamma$ is the image of the canonical map $H\longrightarrow H^*$ defined by $h\longmapsto \digamma(h,.)$.*
(LEA4) *$adx\in End_{\mathbb{F}}(L)$ is locally nilpotent for all $x\in L_\alpha$ and $\alpha\in R^\times$.*

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**2000 Mathematics Subject Classification:** 13D45, 13D07.
**keywords and phrases:** Extended affine Lie algebras, Lie tori and locally extended affine Lie algebras.

(LEA5) $R^\times$ *is irreducible. That is,* $R^\times = R_1 \cup R_2$ *and* $(R_1, R_2) = 0$ *imply that* $R_1 = \phi$ *or* $R_2 = \phi$.

It is showed in [2] that the core of any LEALA is a general version of Lie torus of type $(\Delta, \Lambda)$, with $\Delta$ a locally finite irreducible root system and $\Lambda$ a free abelian group of finite rank. We will now describe a construction of LEALA's, which starts from any such version of centreless Lie tori.

## 2. Main results

Our construction uses data $(\mathcal{L}, D, C, \tau)$, as in [4], described below.

Let $\mathcal{L} = \bigoplus_{\alpha\in\Delta}\bigoplus_{\lambda\in\Lambda}\mathcal{L}_\alpha^\lambda$ be a centreless $(\langle\Delta\rangle\oplus\Lambda)-$ graded Lie algebra with $\Delta$ a locally finite irreducible root system and $\Lambda$ a torsion free abelian group, which satisfies axioms $(LT1)-(LT4)$ described in [4]. It thus follows from [5, Theorems 2.2 and 7.1] that $\mathcal{L}$ has a non-zero nondegenerate invariant symmetric bilinear form $(\cdot,\cdot)$, which is $\Lambda-$graded. Since $\text{supp}_{\langle\Delta\rangle}\mathcal{L}$ is a subsystem of $\Delta$, we can assume $\text{supp}_{\langle\Delta\rangle}\mathcal{L} = \Delta$, and that $\Lambda$ is spanned by $\text{supp}_\Lambda\mathcal{L} = \bigcup_{\alpha\in\Delta}\Lambda_\alpha$, where $\Lambda_\alpha = \{\lambda\in\Lambda \ : \ \mathcal{L}_\alpha^\lambda \neq 0\}$. We can choose families $(x_\alpha^\lambda, h_\alpha^\lambda, y_\alpha^\lambda) \in \mathcal{L}_\alpha^\lambda\times\mathcal{L}_0^0\times\mathcal{L}_{-\alpha}^{-\lambda}$, $\alpha\in\Delta$ and $\lambda\in\Lambda_\alpha$, by $(LT3)$. In particular, the triple $(x_\alpha^\lambda, h_\alpha^\lambda, y_\alpha^\lambda)$ is an $\mathfrak{sl}_2-$ triplet for $\alpha\in\Delta^\times$. We let $\mathfrak{h} = \text{span}_\mathbb{F}\{h_\alpha^0 \ : \ \alpha\in\Delta_{ind}^\times\}$. Then $\mathfrak{h}$ is a toral subalgebra of $\mathcal{L}$ and $\Delta$ canonically embeds into the dual space $H^*$ such that $<\alpha, \check{\beta}>= \alpha(h_\alpha^0)$ for $\alpha\in\Delta_{ind}$ and $\beta\in\Delta_{ind}^\times$. The root spaces of $(\mathcal{L},\mathfrak{h})$ are the subspaces of $\mathcal{L}_\alpha$, $\alpha\in\Delta$.

Recall that the centroid of $\mathcal{L}$, denoted by $\mathcal{C}(\mathcal{L})$, is the set of all $\phi\in End_\mathbb{F}(\mathcal{L})$ satisfying $[\chi, adx] = 0$ for all $x\in\mathcal{L}$. It follows from [1, Lemma 3.4] that

$$\mathcal{C}(\mathcal{L}) = \bigoplus_{\lambda\in\Lambda}\mathcal{C}(\mathcal{L})^\lambda, \tag{1}$$

where $\mathcal{C}(\mathcal{L})^\lambda$ consists of all endomorphisms of degree $\lambda$ with respect to $\Lambda-$grading of $\mathcal{L}$. Let

$$\Gamma = \{\lambda\in\Lambda \mid \mathcal{C}(\mathcal{L})^\lambda \neq 0\},$$

which is a subgroup of $\Lambda$, called the centroidal grading group. We denote by

$$grSCDer_\mathbb{F}(\mathcal{L}) = \bigoplus_{\gamma\in\Gamma}(SCDer_\mathbb{F}(\mathcal{L}))^\gamma,$$

the $\Gamma-$graded subalgebra of centroidal derivations which are skew-symmetric with respect to the $\Lambda-$graded form. Let

$$\mathcal{D} := \{\delta_\theta \ : \ \theta\in Hom_\mathbb{Z}(\Lambda,\mathbb{F})\},$$

be the so-called degree derivations of $\mathcal{L}$, which is contained in $(SCDer_\mathbb{F}(\mathcal{L}))^0$. Since $\Lambda$ is torsion-free, there exists an embedding of the abelian group $\Lambda$ into $\mathcal{D}^*$, that is

$$\text{ev} \ : \ \Lambda \longrightarrow \mathcal{D}^* \quad ; \quad \lambda\longmapsto \text{ev}_\lambda \quad \text{where} \quad \text{ev}_\lambda(\delta_\theta) = \theta(\lambda).$$

The second ingredient of our construction is a $\Gamma-$graded subalgebra of the $grSCDer_\mathbb{F}(\mathcal{L})$,

$$D = \bigoplus_{\gamma\in\Gamma} D^\gamma \quad , \quad D^\gamma \subseteq (SCDer_\mathbb{F}(\mathcal{L}))^\gamma,$$

which has the property that $D^0$ induces the $\Lambda-$grading of $\mathcal{L}=\bigoplus_{\lambda\in\Lambda}\mathcal{L}^\lambda$ where $\mathcal{L}^\lambda=\{x\in\mathcal{L}\ :\ \delta_\theta(x)=\theta(\lambda)x$ for all $\delta_\theta\in D^0\}$. Equivalently, the restricted evaluation map

$$\Lambda\longrightarrow(D^0)^*\quad;\quad\lambda\longmapsto \mathrm{ev}_\lambda\mid_{D^0},$$

is injective.

Let $grD^*=\bigoplus_{\gamma\in\Gamma}(D^*)^\gamma$ be the graded dual space of $D$, thus, $f\in(D^*)^\gamma$ is extended to a linear form on $D$ by $f\mid_{D^\eta}=0$ for $\gamma\neq\eta$.

The third ingredient of our construction is a $\Gamma-$graded subspace of $grD^*$ denoted by $C=\bigoplus_{\gamma\in\Gamma}C^\gamma$, which is invariant under the contragradient action of $D$ on $grD^*$ and contains $\mathrm{span}_{\mathbb{F}}\{\sigma_D(x,y)\ :\ x,\ y\in\mathcal{L}\}$, where

$$\sigma_D\ :\ \mathcal{L}\times\mathcal{L}\longrightarrow grD^*,$$

is the central 2-cocycle for $\mathcal{L}$ (i.e with values in the trivial $\mathcal{L}-$module $grD^*$ which respects the grading of $\mathcal{L}$ and $grD^*$), defined by

$$\sigma_D(x,y)(d)=(d(x),y),\qquad(x,\ y\in\mathcal{L}\quad\text{and}\quad d\in D).$$

Note that, $C^0$ is a subspace such that the restriction map $C^0\longrightarrow(D^0)^*$ is injective and $\sigma_D(x^\lambda_\alpha,y^\lambda_\alpha)\in C^0$ for all $\alpha\in\Delta,\ \lambda\in\Lambda_\alpha$.

The forth ingredient of our construction is an invariant toral $2-$cocycle $\tau\ :\ D\times D\longrightarrow C$. Namely, by considering a $D-$module via the contragredient action denoted by $d\cdot f$ for $d\in D,\ f\in(D^*)^\gamma$, $\tau$ is the bilinear map such that for $d,\ d_i\in D$ $(i=1,\ 2,\ 3)$, satisfies the following:

$$\begin{aligned}\tau(d,d)&=0,\\ \sum_{\circlearrowleft}\tau([d_1,d_2],d_3)&=\sum_{\circlearrowleft}d_1\cdot\tau(d_2,d_3),\\ \tau(d_1,d_2)(d_3)&=\tau(d_2,d_3)(d_1),\\ \tau(D^0,D)&=0.\end{aligned}$$

Here $\sum_{\circlearrowleft}$ indicates the sum over all cyclic permotations of $(1,2,3)$.

Finally, for $\mathcal{L},\ D,\ C$ and $\tau$ as above we define

$$L=L(\mathcal{L},D,C,\tau)=\mathcal{L}\oplus C\oplus D.$$

Then $L$ is a Lie algebra with respect to the product

$$\begin{aligned}[x_1\oplus f_1\oplus d_1,x_2\oplus f_2\oplus d_2]&=([x_1,x_2]+d_1(x_2)-d_2(x_1))\\ &\oplus(\sigma_D(x_1,x_2)+d_1\cdot f_2-d_2\cdot f_1+\tau(d_1,d_2))\oplus[d_1,d_2],\end{aligned}$$

where $x_i\in\mathcal{L},\ f_i\in C$ and $d\in D$ $(i=1,\ 2)$. $L$ has a nondegenerate invariant form, denoted by $(.|.)$ given by

$$(x_1\oplus f_1\oplus d_1|x_2\oplus f_2\oplus d_2)=(x_1,x_2)+f_1(d_2)+f_2(d_1).$$

Thus, $(LEA2)$ holds for $L$. Let $H=\mathfrak{h}\oplus C^0\oplus D^0$. We identify $\Lambda=\mathrm{ev}(\Lambda)\subseteq(D^0)^*$ and view $\Lambda\subseteq H^*$ by letting $\lambda\in\Lambda$ act by zero on $\mathfrak{h}\oplus C^0$. Similarly, any $\alpha\in\Delta\subseteq\mathfrak{h}^*$ gives rise to a bilinear form on $H$ by putting $\alpha\mid_{C^0\oplus D^0}=0$. With these identifications, $H$ becomes a toral subalgebra of $L$ which is self-centralizing and ad-diagonalizable, that is, $(LEA1)$ holds. To describe the roots of toral pair $(L, H)$. For $\alpha \in \Delta$ and $\lambda \in \Lambda_\alpha$ we define a linear form $\alpha + \lambda \in H^*$ by

$$(\alpha + \lambda)(h \oplus f \oplus d) = \alpha(h) + \mathrm{ev}_\lambda(d),$$

where $h \in \mathfrak{h},\ f \in C^0$ and $d \in D^0$. Then the root spaces of toral pair $(L, H)$ are

$$L_{\alpha+\lambda} = \begin{cases} \mathcal{L}_\alpha^\lambda & ; \alpha \neq 0,\ \lambda \in \Lambda_\alpha \\ \mathcal{L}_0^\lambda \oplus C^\lambda \oplus D^\lambda & ; \alpha = 0,\ \lambda \in \Lambda_0. \end{cases} \tag{2}$$

Thus, the set of roots of $(L, H)$ is

$$R = \{\alpha + \lambda \ :\ \alpha \in \Delta,\ \lambda \in \Lambda_\alpha\} = \bigcup_{\alpha \in \Delta} (\alpha \oplus \Lambda_\alpha) \subseteq H^*.$$

So, $R$ is contained in the image $H^*$ of $H$ with respect to the form $(.|.)$, that is, the axiom $(LEA3)$ is true for $(L, H)$. By construction, $H$ is a splitting cartan subalgebra, so (LEA5) holds.

The above construction is summarized in the following.

**Theorem 2.1.** *The pair $(L, H)$, where $L = L(\mathcal{L}, D, C, \tau)$, is a LEALA with respect to the form $(.|.)$. Moreover, its centreless core is $\mathcal{L}$.*

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