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SOME RECENT APPLICATIONS OF THE HOMOTOPY ANALYSIS METHOD

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ABSTRACT. Here, the homotopy analysis method (HAM), which is a powerful and easy-to-use analytic tool for nonlinear problems and does not need small parameters in the equations, is introduced. The homotopy analysis method contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series, but it is heuristic rather than a mathematical method.

1. Introduction

Most of engineering problems, especially some heat transfer equations are nonlinear, and in most cases it is difficult to solve them, especially analytically. Perturbation method is one of the well-known methods to solve nonlinear problems. It is based on the existence of small/large parameters, the so-called perturbation quantity [1, 2]. Many nonlinear problems do not contain such kind of perturbation quantity, and we can use non-perturbation methods, such as the artificial small parameter method [3], the δ-expansion method [4], the Adomian's decomposition method [5]. However, both of the perturbation and non-perturbation methods cannot provide us with a simple way to adjust and control the convergence region and rate of given approximate series.

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In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely *Homotopy Analysis Method* (HAM), [6, 7]. This method has been successfully applied to solve many types of nonlinear problems by others.

In this talk, the basic idea of the HAM is introduced and then its application in some heat transfer equations is studied. In addition, the nonlinear equation of the heat radiation is solved through HAM as an example.

2. **Basic idea of HAM**

To show the basic idea, let us consider the following differential equation

$$\mathcal{N}[u(\tau)] = 0,$$

where $\mathcal{N}$ is a nonlinear operator, τ denotes the independent variable and $u(\tau)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [7] constructs the so-called zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(\tau;p) - u_0(\tau)] = p\hbar H(\tau)\mathcal{N}[\phi(\tau;p)], \tag{1}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a non-zero auxiliary parameter, $H(\tau) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, $\phi(\tau;p)$ is an unknown function, respectively.

It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, we have

$$\phi(\tau;0) = u_0(\tau), \quad \phi(\tau;1) = u(\tau),$$

respectively. Thus as p increases from 0 to 1, the solution $\phi(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. By expanding $\phi(\tau;p)$ in Taylor series with respect to p, one has

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)p^m, \tag{2}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau;p)}{\partial p^m}\Big|_{p=0}. \tag{3}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, then the series (2) converges at $p = 1$, and we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau),$$

which must be one of the solutions of original nonlinear equation, as proved by Liao [7]. If $\hbar = -1$ and $H(\tau) = 1$ then Eq. (1) becomes

$$(1-p)\mathcal{L}[\phi(\tau;p) - u_0(\tau)] + p\mathcal{N}[\phi(\tau;p)] = 0,$$

which is used mostly in the homotopy perturbation method (HPM), whereas we have obtained the solution directly, *without using Taylor series*. The comparison between HAM and HPM can be found in [8].

According to the definition (3), the governing equation can be deduced from the zero-order deformation equation (1). Define the vector $\overrightarrow{u}_n = \{u_0(\tau), u_1(\tau), \cdots, u_n(\tau)\}$. Differentiating Eq.(1) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}(u_m(\tau) - \chi_m u_{m-1}(\tau)) = \hbar H(\tau) R_m(\overrightarrow{u}_{m-1}), \tag{4}$$

where

$$R_m(\overrightarrow{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(\tau;p)]}{\partial p^{m-1}}\Big|_{p=0}, \tag{5}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

3. **A practical example**

In this part, we consider the following example which is used in studying the cooling of a lumped system with variable specific heat

$$(1+\epsilon u)\frac{\mathrm{d}u}{\mathrm{d}\tau} + u = 0, \quad u(0) = 1. \tag{6}$$

According to the governing equation and the initial condition (6), the solution can be expressed by a set of base functions $\{e^{-n\tau} | n =$

$1, 2, 3, \ldots\}$, in the form

$$u(\tau) = \sum_{n=1}^{+\infty} d_n e^{-n\tau}, \tag{7}$$

where d_n is a coefficient to be determined. This provides us with the so-called *rule of solution expression*, i.e. the solution of (6) must be expressed in the same form as (7) and the other expressions such as $\tau^m e^{-n\tau}$ must be avoided.

According to (6) and (7), we choose the linear operator $\mathcal{L}[\phi(\tau;p)] = \frac{\partial\phi(\tau;p)}{\partial\tau} + \phi(\tau;p)$, with the property $\mathcal{L}[c_1 e^{-\tau}] = 0$, where c_1 is constant. From (6), we define a nonlinear operator

$$\mathcal{N}[\phi(\tau;p)] = (1 + \epsilon\phi(\tau;p))\frac{\partial\phi(\tau;p)}{\partial\tau} + \phi(\tau;p). \tag{8}$$

According to (6) and the rule of solution expression (7), it is straightforward that the initial approximation should be in the form $u_0(\tau) = e^{-\tau}$, and the initial condition of the zero-order deformation equation (1) is $\phi(0;p) = 1$.

From Eq. (5) and (8), we have

$$R_m(\overrightarrow{u}_{m-1}) = u'_{m-1}(\tau) + \epsilon \sum_{n=0}^{m-1} u_n(\tau) u'_{m-1-n}(\tau) + u_{m-1}(\tau),$$

where the prime denotes differentiation with respect to the similarity variable τ. Now, the solution of the mth-order deformation Eq. (4) for $m \geq 1$ becomes

$$u_m(\tau) = \chi_m u_{m-1}(\tau) + \hbar e^{-\tau} \int_0^{\tau} e^{\eta} H(\eta) R_m(\overrightarrow{u}_{m-1}) \mathrm{d}\eta + c_1 e^{-\tau},$$

where the integral constant c_1 is determined by the initial condition of (6).

According to the rule of solution expression denoted by (7) and from Eq. (4), the auxiliary function $H(\tau)$ should be in the form $H(\tau) = e^{-\kappa\tau}$, where κ is an integer. It is found that, when $\kappa \leq -1$, the solution of the high-order deformation Eq.(4) contains the term $\tau e^{-\tau}$, which incidentally disobeys the rule of solution expression (7). When $\kappa \geq 1$, the base $e^{-2\tau}$ always disappears in the solution expression of the high-order deformation Eq.(4), so that the coefficient of the term $e^{-2\tau}$ cannot be modified even if the order of approximation tends to infinity. This rule, so-called *rule of coefficient ergodicity* by Liao [7]. Thus, we had to set $\kappa = 0$, which uniquely determines the corresponding auxiliary function $H(\tau) = 1$.

Therefore, we have

$$\begin{aligned} u_1(\tau) &= e^{-\tau}(-\epsilon\hbar) + e^{-2\tau}(\epsilon\hbar), \\ u_2(\tau) &= e^{-\tau}(-\epsilon\hbar(1+\hbar) + \frac{\epsilon^2}{2}\hbar^2) \\ &\quad + e^{-2\tau}(\epsilon\hbar(1+\hbar) - 2\epsilon^2\hbar^2) + e^{-3\tau}(\frac{3}{2}\epsilon^2\hbar^2), \end{aligned}$$

which are as the same solutions obtained by perturbation method and HPM, by setting $\hbar = -1$.

The exact solution of (6), by separating the variables can be obtained as

$$\ln u + \epsilon(u-1) + \tau = 0.$$

Hence, the mth-order approximation of $u(\tau)$ can be generally expressed by

$$u(\tau) \approx \sum_{n=1}^{m+1} \gamma_{m,n}(\hbar) e^{-n\tau}, \tag{9}$$

where $\gamma_{m,n}(\hbar)$ is a coefficient dependent of $\hbar$. Equation (9) is a family of solution expression in the auxiliary parameter $\hbar$. For considering the influence of $\hbar$ on the convergence of (9), we first plot the so-called $\hbar$-curves of $u'(0)$ and $u''(0)$. As shown in Figure 1, it is easy to discover the valid region of $\hbar$.

We can see that the best value for $\hbar$ is not -1 and it depends on ϵ, i.e. for $\epsilon = 0.5$ and $\hbar = -0.9$, or for $\epsilon = 1$ and $\hbar = -0.8$, HAM gives better solution. In Figure 2, we see that it is better to take another values for $\hbar$ except -1, especially for large ϵ.

4. Conclusions

In this talk, the homotopy analysis method (HAM) is applied to obtain the solution of nonlinear equations arising in heat transfer. HAM provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between HAM and other methods. Also, it has been shown that the HPM and perturbation method are valid only for small parameter ϵ, but in HAM we can choose $\hbar$ in appropriate way. This work shows the validity and great potential of the HAM for nonlinear problems in science and engineering.

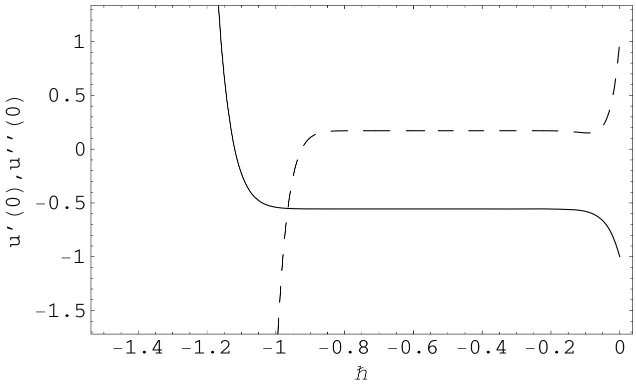

Figure 1 : The $\hbar$-curves for $\epsilon = 0.8$, solid line: 15th-order approximation of $u'(0)$; dashed line: 15th-order approximation of $u''(0)$.

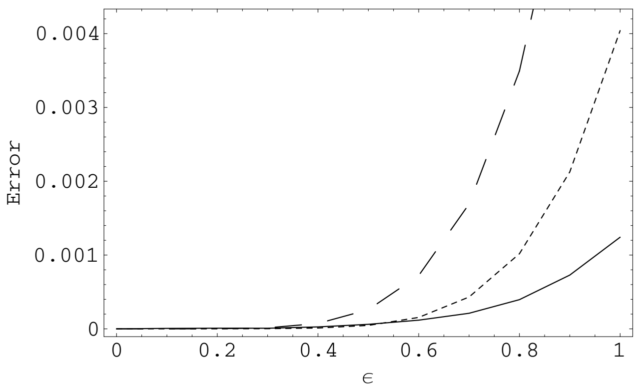

Figure 2 : Error of HAM with respect to ϵ for various $\hbar$ by 5th-order approximation of solution, solid line: $\hbar = -0.8$; dotted line: $\hbar = -0.9$; dashed line: $\hbar = -1$.

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