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TYPES OF DERIVATIONS

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ABSTRACT. Let A be an algebra. A linear mapping $\delta : A \to A$ is called a *derivation* if it satisfies the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$. In this talk we introduce some notions related to this concept and we propose some problems concerning these notions.

1. Preliminaries

Let A be an algebras. A linear mapping $\delta : A \to A$ is called a *derivation* if it satisfies the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$. For a fixed element a_0 of A, the mapping $\delta_{a_0} : A \to A$ defined by $\delta_{a_0}(a) = a_0a - aa_0$ is called the *inner derivation implemented by* a_0. One of the most important questions about derivations is to determine algebras A in which each derivation is inner implemented by an element of the algebra. Sometimes we can find the element a_0 in an algebra B containing A. In many cases, such an element can be found in the closure of A in B with respect to a certain topology on B. This is the reason that we are interested in *approximately inner derivations*. This causes the problem of automatic continuity of derivations in algebras which are equipped with a topology and among which we are interested in normed algebras. Innerness and automatic continuity are two nice problems in the realm of derivations. The former is a purely algebraic problem and the latter is interesting in analysis, though, an affirmative

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answer for the latter provides some approach for the first problem. In this talk, we introduce some notions related to the concept of a derivation and examine these two problems in each case.

2. **Related Notions**

Let A be an algebra and $\sigma : A \to A$ be a morphism. If $\delta : A \to A$ is a derivation, then $d = \delta\sigma$ satisfies the equation

$$d(ab) = \delta\sigma(ab) = \delta(\sigma(a)\sigma(b)) = d(a)\sigma(b) + \sigma(a)d(b)$$

for all $a, b \in A$. This motivates us to consider the following definition.

Definition 2.1. *Let A be an algebra and $\sigma, \tau : A \to A$ be two linear mappings. A linear mapping $d : A \to A$ is called a (σ, τ)-derivation if*

$$d(ab) = d(a)\sigma(b) + \tau(a)d(b)$$

for all $a, b \in A$.

There are some questions here:

- Let d be a (σ, τ)-derivation. Can we deduce that both σ and τ are morphisms?
- Is it true that a (σ, σ)-derivation d is $\delta\sigma$ for some derivation δ?
- Is there any relation between the automatic continuity of a (σ, τ)-derivation d and the automatic continuity of σ and τ?

There are some affirmative answers to the above questions in certain cases (see [6] and [7]):

Theorem 2.2. *If σ and τ are continuous $*$-linear mappings from a C^*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathfrak{H}$ into $B(\mathfrak{H})$, then every $*$-(σ, τ)-derivation $d : \mathfrak{A} \to B(\mathfrak{H})$ is automatically continuous.*

Theorem 2.3. *Let σ be an ultraweakly continuous surjective $*$-linear mapping on a von Neumann algebra $\mathfrak{M}$. Then every σ-derivation $d : \mathfrak{M} \to \mathfrak{M}$ is automatically ultraweakly continuous.*

Moreover, the following question is also interesting:

- Let $\sigma, \tau : A \to A$ be two linear mappings. Can we extend the definition of an inner derivation to a (σ, τ)-derivation?

A natural intuition tempts us to consider $d(a) = a_0\sigma(a) - \tau(a)a_0$ as the *inner (σ, τ)-derivation implemented by a_0* (see [2]), but this works when σ and τ are morphisms or at least we have

$$a_0(\sigma(ab) - \sigma(a)\sigma(b)) = (\tau(ab) - \tau(a)\tau(b))a_0.$$

Some results about innerness of (σ,τ)-derivations are in [5]:

Theorem 2.4. *Suppose that $\mathfrak{M}$ is a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ and $\sigma:\mathfrak{M}\to\mathfrak{M}$ is an ultraweakly continuous surjective $*$-linear mapping and $d:\mathfrak{M}\to\mathfrak{M}$ is an ultraweakly continuous $*$-σ-derivation such that $d(I)$ is a central element of $\mathfrak{M}$. Then $\mathfrak{H}$ can be decomposed into $\mathfrak{K}\oplus\mathfrak{L}$ and d can be factored as the form $\delta\oplus 2Z\tau$,where $\delta:\mathfrak{M}\to\mathfrak{M}$ is an inner $*$-$\sigma_{\mathfrak{K}}$-derivation, Z is a central element, and $2\tau=2\sigma_{\mathfrak{L}}$ is a $*$-homomorphism.*

In another direction, we can consider two derivations δ_1 and δ_2 on A and define $d:A\to A$ by $d=\delta_1\delta_2$. This linear mapping satisfies

$$d(a)=\delta_1(\delta_2(a)b+a\delta_2(b))=d(a)b+\delta_1(a)\delta_2(b)+\delta_2(a)\delta_1(b)+ad(b)$$

for all $a,b\in A$. This is our motivation for the following definition.

Definition 2.5. *Let A be an algebra and $\delta_1,\delta_2:A\to A$ be two linear mappings. A linear mapping $d:A\to A$ is called a (δ_1,δ_2)-double derivation if*

$$d(ab)=d(a)b+\delta_1(a)\delta_2(b)+\delta_2(a)\delta_1(b)+ad(b)$$

for all $a,b\in A$.

We have similar questions again:

- Let d be a (δ_1,δ_2)-double derivation. Can we deduce that both δ_1 and δ_2 are derivations?
- Is it true that a (δ,δ)-derivation d is δ^2 for some derivation δ?
- Is there any relation between the automatic continuity of a (δ_1,δ_2)-double derivation d and the automatic continuity of δ_1 and δ_2?

We have affirmative answers in certain cases in [9].

Definition 2.6. *Let A be an algebra. A sequence $d_n:A\to A$ of linear mappings is called a higher derivation if*

$$d_n(ab)=\sum_{k=0}^{n}d_k(a)d_{n-k}(b)$$

for each $a,b\in A$ and each nonnegative integer n.

If δ is a derivation, then $d_n=\frac{\delta^n}{n!}$ is a higher derivation which is called an *ordinary higher derivation*. We are now interested in the following questions:

- Is there any example of a higher derivation which is not ordinary?
- Is there a characterization of higher derivations in terms of derivations defined on the algebra A?
- Is there any relation between the automatic continuity of higher derivations on an algebra and the automatic continuity of its derivations?

We have the following nice result in [3].

Theorem 2.7. *Let $\{d_n\}$ be a higher derivation on an algebra A with $d_0 = I$. Then there is a sequence $\{\delta_n\}$ of derivations on A such that*

$$d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \ldots + r_i} \right) \delta_{r_1} \ldots \delta_{r_i} \right),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$.

Higher derivations were introduced by Hasse and Schmidt [1], and algebraists sometimes call them Hasse-Schmidt derivations. A notion of an inner higher derivation is given in [10].

The above notion leads us to the following two new concepts.

Definition 2.8. *Let A be an algebra. We say that a sequence $\{d_n\}$ of linear mappings on A is a prime higher derivation if $d_n(ab) = \sum_{k|n} d_k(a) d_{\frac{n}{k}}(b)$ for each $a, b \in A$ and each $n \in \mathbb{N}$.*

We have the same question for prime higher derivations as we have for higher derivations which is considered in [4].

Definition 2.9. *Let A be an algebra and n be a fixed positive integer. A linear mapping $d : A \to A$ is called a power derivation of order n if $d^n(ab) = \sum_{k=0}^{n} \binom{n}{k} d^k(a) d^{n-k}(b)$ for each $a, b \in A$.*

The following questions for power derivations are considered in [8]:

- Is there any example of a power derivation of order n which is not of order $n - 1$?
- If d is a power derivation of two consecutive orders n and $n + 1$, then can we deduce that d is a power derivation of order m for each $m \geq n$?

References

[1] H. Hasse and F. K. Schmidt, *Noch eine Begrüdung der theorie der höheren Differential quotienten in einem algebraischen Funtionenkörper einer Unbestimmeten*, J. Reine Angew. Math., **177** (1937), 215-237.

[2] H. Komatsu, On inner (σ, τ)-derivations, *Math. J. Okayama Univ.*, **23(1)** (1981), 33-36.

[3] M. Mirzavaziri, *Characterization of higher derivations on algebras*, Communications in Algebra (to appear).

[4] M. Mirzavaziri, *Prime higher derivations on algebras*, Bull. Iranian Math. Soc. (to appear).

[5] M. Mirzavaziri and M. S. Moslehian, *A Kadison-Sakai type theorem*, Bull. Austral. Math. Soc. (to appear).

[6] M. Mirzavaziri and M. S. Moslehian, *Automatic continuity of σ-derivations in C^*-algebras*, Proc. Amer. Math. Soc., **11** (2006), no. 5, 805–813.

[7] M. Mirzavaziri and M. S. Moslehian, *Ultraweak continuity of σ-derivations on von Neumann algebras*, Math. Phys. Anal. Geom. (to appear).

[8] M. Mirzavaziri, K. Naranjani and A. Niknam, *Power derivations* (preprint).

[9] M. Mirzavaziri and E. Omidvar Tehrani, *δ-double derivations on C^*-algebras*, Bull. Iranian Math. Soc. (to appear).

[10] A. Roy and R. Sridharan, *Higher derivations and central simple algebras*, Nagoya Math. J., **32** (1968), 21–30.