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APPLICATION OF PRE-SCHWARZIAN DERIVATIVE TO CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract. In this paper, we give a brief survey of the development of areas of mathematics related to geometric function theory. We are mainly concerned with univalent functions. Also, we introduce some results involving pre-Shwarzian derivatives.

1. **Introduction**

Let $\mathcal{A}$ be the linear space of all analytic functions in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$.

By $S, S^*(\alpha)$ and $K(\alpha)$ we denote the well known subclasses of $\mathcal{A}$ that are univalent, starlike (with respect to the origin) of order α, convex of order α, respectively.

Note that $f \in S^*(\alpha) \Leftrightarrow J[f] \in K(\alpha)$, where $J[f]$ denotes the Alexander transform [1] of $\mathcal{A}$ defined by

$$J[f](z) = \int_0^z \frac{f(\xi)}{\xi} d\xi = \int_0^1 \frac{f(tz)}{t} dt.$$

In 1960, Bienacki claimed that $f \in S$ implies $J[f] \in S$, but this turned out to be wrong (see[4]). This means that the Alexander integral operator does not preserve the class S. In [5] Y. J. Kim and Merkes

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considered the nonlinear integral transform J_α, defined by

$$J_\alpha[f](z)=\int_0^z\left(\frac{f(t)}{t}\right)^\alpha dt$$

for the complex number α and for functions f in the class

$$\mathcal{Z}f=\{f\in\mathcal{A}: f(z)\neq 0 \quad forall \quad 0<|z|<1\},$$

and showed that

$$J_\alpha(S)=\{J_\alpha[f]: f\in S\}\subset S.$$

Up to now, the best constant is not known for this result.

Also let $\mathcal{U}$ denote the class of uniformly locally univalent, that is, there exists a constant $\rho=\rho(f)>0$ such that f is univalent in each disk of hyperbolic radius ρ in D.

Let $f: D\longrightarrow\mathbb{C}$ be analytic and locally univalent. The pre-Schwarzian derivative T_f of f is defined by $T_f(z)=\frac{f''(z)}{f'(z)}$. Also, with respect to the Hornich operation, the quantity

$$||T_f||=\sup_{z\in D}(1-|z|^2)|T_f(z)|$$

can be regarded as a norm on the space of uniformly locally univalent analytic functions f in D. It is known that f is uniformly locally univalent if and only if $||T_f||<\infty$. In connection with the above norm, the following result is important to note.

Theorem 1.1. *([2],[3],[6]) Let f be analytic and locally univalent in D. Then*

i) if $||T_f||\leq 1$ then f is univalent, and

ii) if $||T_f||\leq 2$ then f is bounded.

The constants are sharp.

We say that a function $\phi\in\mathcal{A}$ is subordinate to $\psi\in\mathcal{A}$ and write $\phi\prec\psi$ or $\phi(z)\prec\psi(z)$ if there is a Schwarz function ω (i.e. a function $\omega\in\mathcal{A}$ with $|\omega(z)|<1$ in D) satisfying $\phi(z)=\psi(\omega(z))$.

The following always generates a sharp result for a fixed g.

Theorem 1.2. *Let $g\in\mathcal{U}$ be given. For an analytic function f on the unit disk D, if f' is subordinate to g' then we have $||T_f||\leq||T_g||$. In particular, f is uinformly locally univalent on the unit disk.*

2. Norm estimate for certain subclasses of univalent functions

As a typical application of Theorem 1.2, we present the following:

Theorem 2.1. *If* $f \in \mathcal{A}$ *satisfies* $Ref'(z) > 0$ *on the unit disk, then* $||T_f|| \leq 2$. *Moreover, the bound is sharp.*

We note that the Noshiro-Warschawski theorem shows that such an f must be univalent.

For a constant $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, a function $f \in \mathcal{A}$ is called β-spiral-like if f is univalent on D and for any $z \in D$, the β-logarithmic spiral $\{f(z)exp(-e^{i\beta}t) : t \geq 0\}$ is contained in $f(D)$. It is equivalent to the condition that

$$Re\left\{e^{-i\beta}\frac{zf'(z)}{f(z)}\right\} > 0 \qquad z \in D.$$

We denote by $Sp(\beta)$ the set of β-spiral-like function.

In the following theorem, Y.Okuyama [9] gives the best estimate of the norms of pre-Schwarzian derivative for the class $Sp(\beta)$.

Theorem 2.2. *For every* $f \in Sp(\beta)$, *where* $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, *we have the following:*

I) In the case $|\beta| \leq \frac{\pi}{3}$ *we have*

$$||T_f|| \leq ||T_{f_\beta}|| = 2|2 + e^{2i\beta}|.$$

II) In the case $|\beta| > \frac{\pi}{3}$ *we have*

$$||T_f|| \leq ||T_{f_\beta}||,$$

where

$$||T_{f_\beta}|| = \max_{0 \leq m \leq \frac{4}{3}sin|\beta|} 2m\cos\beta\left(1 + \sqrt{\frac{m^2 + 4 - 4m\sin|\beta|}{m^2 + 1 - 2msin|\beta|}}\right),$$

and $f_\beta(z) = z(1-z)^{-2e^{i\beta}\cos\beta} \in Sp(\beta)$ *is the* β*-spiral Koebe function. In both cases, the equality* $||T_f|| = ||T_{f_\beta}||$ *holds if and only if* f *is a rotation of the* β*-spiral Koebe function.*

Let α be a real constant with $0 \leq \alpha \leq 1$. An analytic function f on the unit disk satisfying $f(0) = 0$ and $f'(0) = 1$ is called strongly starlike of order α if f satisfies the condition

$$|\arg\frac{zf'(z)}{f(z)}| \leq \frac{\alpha\pi}{2} \qquad (z \in D^* = D\backslash\{0\}).$$

T. Sugawa in [8] has proved the following result for strongly starlike functions.

Theorem 2.3. *If f is strongly starlike of order α, then* $||T_f|| \leq M(\alpha) + 2\alpha$, *where,* $M(\alpha)$ *is given by*

$$M(\alpha) = \frac{4\alpha c(\alpha)}{(1-\alpha)c(\alpha)^2 + 1 + \alpha}. \tag{1}$$

and $c(\alpha)$ is the unique solution of the following equation with respect to x in the interval $(1, \infty)$;

$$(1-\alpha)x^{\alpha+2} + (1+\alpha)x^{\alpha} - x^2 - 1 = 0$$

Moreover, the equality holds precisely if $T_f(z) = (\frac{1+\epsilon z}{1-\epsilon z})^{\alpha}$ *for a constant* ϵ *with* $|\epsilon| = 1$.

Also we have the following result for starlike and convex functions of order α, which is due to Yamashita [10].

Theorem 2.4. *Let* $0 \leq \alpha < 1$ *and* $f \in S$.

1) If f is starlike of order α, i.e, $Re(zf'(z)/f(z)) > \alpha$, *then* $||T_f|| \leq 6 - 4\alpha$.

2) If f is convex of order α, i.e, $Re(1 + zf''(z)/f'(z)) > \alpha$, *then* $||T_f|| \leq 4(1-\alpha)$.

Moreover, the above bounds are sharp.

3. Application of pre-Schwarzian derivative

For a non-negative λ we set $B(\lambda) = \{f \in \mathcal{A} : ||T_f|| \leq 2\lambda\}$. The following theorem is significant in connection with univalent functions.

Theorem 3.1. *(Becker and Pommerenk [2],[3]) The set S of normalized univalent analytic functions on the unit disk is contained in $B(3)$ and contains $B(\frac{1}{2})$. The result is sharp.*

In the class $B(\lambda)$ for $0 \leq \lambda < \infty$, the function $F_\lambda(z) = \int_0^z (\frac{1+t}{1-t})^{\lambda} dt$ is extremal, so it is important to know the univalence of F_λ.

Theorem 3.2. *([6]). For a non-negative number λ, the function F_λ is univalent in the unit disk if and only if* $0 \leq \lambda \leq 1$.

Theorem 3.3. *([6]) For $\lambda > 1$ every $f \in B(\lambda)$ satisfies the growth condition*

$$f(z) = O(1 - |z|)^{1-\lambda}$$

as $|z| \to 1$. *On the other hand , for* $\lambda < 1$, *every function* $f \in B(\lambda)$ *is bounded with the uniform bound* F_λ.

Theorem 3.4. *([6]) Let* $f(z) = z + a_2z^2 + ... \in S$. *If* $||T_f|| \leq 2\lambda$ *with* $1.491 < \lambda \leq 3$, *then* $a_n = O(n^{\lambda-2})$ *as* $n \longrightarrow \infty$. *This order estimate is best possible.*

The function $f(z)$, analytic in the unit disk D, is called a Bloch function if

$$||f|| = |f(0)| + \sup_{z\in D}(1-|z|^2)|f'(z)| < \infty.$$

The linear space of such functions is denoted by B. With the above norm B is a complex Banach space.

Let $BMOA$ be the family of functions f analytic in D with finite $BMOA$ norm

$$||f||_* = \sup ||f_\alpha||_2 + |f(0)| < \infty,$$

where $f_\alpha(z) = f(\frac{z+\alpha}{1+\bar{\alpha}z}) - f(\alpha)$.

The following well known theorem states the relationship between the class B and $BMOA$.

Theorem 3.5. *Let* f *be a univalent analytic function on the unit disk. Then* $f \in BMOA$ *if and only if* f *is a Bloch function.*

The Hardy space $\mathcal{H}^p (0 < p \leq \infty)$ is the class of all functions f analytic in D such that

$$||f||_p = \lim_{r\mapsto 1} M_p(r,f) < \infty,$$

where

$$M_p(r,f) := \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, & 0 \leq p < \infty; \\ \max_{|z|\leq r} |f(z)|, & p = \infty. \end{cases}$$

Note that $\mathcal{H}^\infty \subset BMOA \subset \bigcap_{0<p<\infty} \mathcal{H}^p$.

Theorem 3.6. *([6]) Let* $f \in S$ *and set* $||T_f|| = 2\lambda$.

1) if $\lambda < 1$ *then* $f \in \mathcal{H}^\infty$.

2) if $\lambda > 1$ *then* $f \in \mathcal{H}^p$ *for all* $0 < p < \frac{1}{\lambda-1}$.

3) if $\lambda = 1$ *then* $f \in BMOA$.

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